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Subtraction terms at NNLO

Stefan Weinzierl
Dipartimento di Fisica, Università di Parma
INFN Gruppo Collegato di Parma, 43100 Parma, Italy
E-mail: stefanw@fis.unipr.it

Abstract: Perturbative calculations at next-to-next-to-leading order for multi-particle final states require a method to cancel infrared singularities. I discuss the subtraction method at NNLO. As a concrete example I consider the leading-colour contributions to $e^+e^- \to 2$ jets. This is the simplest example which exhibits all essential features. For this example, explicit subtraction terms are given, which approximate the four-parton and three-parton final states in all double and single unresolved limits, such that the subtracted matrix elements can be integrated numerically. The subtraction terms still have to be integrated analytically over the unresolved phase space.

Keywords: QCD, Jets.

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Contents

1. Introduction

Present and future collider experiments will provide a large sample of multi-particle final states. In order to extract information from this data, precise theoretical calculations are necessary. This implies to extend perturbative calculations from next-to-leading order (NLO) to next-to-next-to-leading order (NNLO). Due to a large variety of interesting jet observables it is desirable not to perform this calculation for a specific observable, but to set up a computer program, which yields predictions for any infra-red safe observable relevant to the process under consideration. Compared to already existing NNLO calculations for specific (inclusive) observables like Drell-Yan or inclusive Higgs production [1]–[5], this requires to work with fully differential cross-sections. This is the major complication of the project. The three major challenges to accomplish this task are: the calculation of two-loop amplitudes, a method for the cancellation of infrared divergences and stable and efficient Monte Carlo techniques.

The last three years have witnessed a tremendous progress in the calculation of two-loop amplitudes [6]–[9]. In particular the two-loop amplitudes for $e^+e^- \to 3$ jets have been calculated [10, 11].

Up to now, less is known for the cancellation of infrared divergences at NNLO. Loop amplitudes, calculated in dimensional regularization, have explicit poles in the dimensional regularization parameter $\varepsilon = 2 - D/2$, arising from infrared singularities. These poles cancel with similar poles arising from amplitudes with additional partons, when integrated over phase space regions where two (or more) partons become "close" to each other. However, the cancellation occurs only after the integration over the unresolved phase space has been performed and prevents thus a naive Monte Carlo approach for a fully exclusive calculation. It is therefore necessary to cancel first analytically all infrared divergences and to use Monte Carlo methods only after this step has been performed. Infrared divergences occur already at next-to-leading order. In this case, general methods to handle the problem are known. Examples are the phase-space slicing method [12, 13] and the subtraction method [14]–[18]. As already mentionend, similar methods at NNLO are not yet known. The major difficulty arises from so-called double unresolved configurations, where three partons become degenerate simultaneously. A phase-space slicing approach to double unresolved configurations has been used in the calculation of the photon + 1-jet rate in electron-positron annihilation [19]. The calculational complexity of this process is somewhere in between a NLO calculation and a full NNLO calculation, as the relevant two-loop amplitude vanishes.

In this paper I discuss the general setup for the subtraction method at NNLO. To illustrate the method, I use as an example the leading-colour NNLO contributions to $e^+e^- \to$ 2 jets. This is the simplest example which exhibits all essential features. For this example I give all necessary subtraction terms. These subtraction terms approximate the leading-colour Born contribution of $e^+e^- \to qgg\bar{q}$ in all double and single unresolved limits, and the leading-colour one-loop contribution of $e^+e^- \to qg\bar{q}$ in all single unresolved limits. The subtracted matrix elements can therefore be integrated numerically over the appropriate phase space. Valuable information for the construction of the subtraction terms is provided by the known behaviour of the tree and one-loop amplitudes in singular limits. The double unresolved limits of tree amplitudes have been considered in [19]–[26]. Single unresolved limits of one-loop amplitudes have been considered in [27]–[33]. However, for the subtraction method this information is not yet sufficient. (It is sufficient for a phase space slicing approach.) Since the subtraction terms are subtracted over the complete phase space, they must have the correct singular behaviour not only in the double unresolved cases, but also in the single unresolved ones. This requires to extend the calculations of the singular behaviour of tree amplitudes and to include also subleading single unresolved singularities.

The method presented here is general and not restricted to the example of $e^+e^- \to$ 2 jets. Subtraction terms for other splittings (like $g \to ggg$) and other kinematical configuartions (e.g. with partons in the initial state) can be worked out along the same lines.

The subtraction terms still need to be integrated analytically over the appropriate unresolved phase space. Although some of the occuring integrals are quite involved, recently developped integration techniques [34, 35] seem to indicate that these integrals can be done analytically.

The knowledge of the subtraction terms will also be useful for an improvement of parton showering algorithms in event generators [36].

This paper is organized as follows: in the following section I discuss the general setup of the subtraction method at NNLO. Section 3 gives the relevant amplitudes for the example $e^+e^- \to$ 2 jets. Section 4 reviews the singular limits of amplitudes. In section 5 I present the NNLO subtraction terms and in section 6 I discuss the mechanism of cancellations between the various terms. Finally section 7 contains the conclusions and an outlook.

2. General idea

In this section I outline the general setup for the subtraction method at NNLO. The NNLO contribution to an observable $\mathcal{O}$ whose LO contribution depends on n partons is given by

$$\begin{aligned} \langle \mathcal{O} \rangle_n^{NNLO} = & \int \mathcal{O}_{n+2}(p_1,\dots,p_{n+2}) d\sigma_{n+2}^{(0)} + \int \mathcal{O}_{n+1}(p_1,\dots,p_{n+1}) d\sigma_{n+1}^{(1)} + \\ & + \int \mathcal{O}_n(p_1,\dots,p_n) d\sigma_n^{(2)} . \end{aligned}$$

The various contributions are symbolically shown in figure 1. The observable $\mathcal{O}$ has to be infrared safe. This requires that whenever a $n+1$ parton configuration p_1,...,p_{n+1} becomes kinematically degenerate with a n parton configuration p'_1,...,p'_n we must have

$$\mathcal{O}_{n+1}(p_1,\dots,p_{n+1}) \to \mathcal{O}_n(p'_1,\dots,p'_n) . \tag{2.1}$$

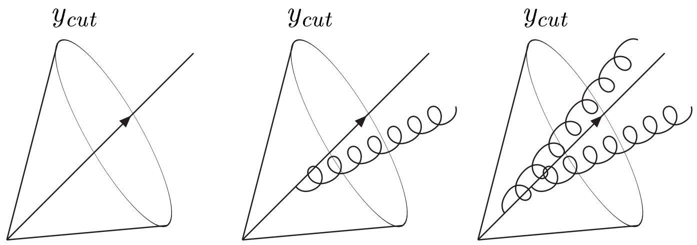

Figure 1: Modelling of jets at next-to-next-to-leading order. The cone labeled with y_{cut} represents the experimental cuts. At next-to-next-to-leading order a jet is modelled either by one, two or three partons.

In addition, we must have in the double unresolved case (e.g. when a $n+2$ parton configuration $p_1,\ldots,p_{n+2}$ becomes kinematically degenerate with a n parton configuration $p'_1,\ldots,p'_n$)

$$\mathcal{O}_{n+2}(p_1,\ldots,p_{n+2}) \to \mathcal{O}_n(p'_1,\ldots,p'_n)\,. \tag{2.2}$$

The subscript n for $\mathcal{O}_n$ and $d\sigma_n^{(l)}$ indicates that this contribution is integrated over a n-parton phase space. The various contributions $d\sigma_n^{(l)}$ are given by

$$\begin{aligned} d\sigma_{n+2}^{(0)} &= \left({\mathcal{A}_{n+2}^{(0)}}^* \mathcal{A}_{n+2}^{(0)} \right) d\phi_{n+2}\,, \\ d\sigma_{n+1}^{(1)} &= \left({\mathcal{A}_{n+1}^{(0)}}^* \mathcal{A}_{n+1}^{(1)} + {\mathcal{A}_{n+1}^{(1)}}^* \mathcal{A}_{n+1}^{(0)} \right) d\phi_{n+1}\,, \\ d\sigma_{n}^{(2)} &= \left({\mathcal{A}_{n}^{(0)}}^* \mathcal{A}_{n}^{(2)} + {\mathcal{A}_{n}^{(2)}}^* \mathcal{A}_{n}^{(0)} + {\mathcal{A}_{n}^{(1)}}^* \mathcal{A}_{n}^{(1)} \right) d\phi_{n}\,, \end{aligned} \tag{2.3}$$

where $\mathcal{A}_n^{(l)}$ denotes an amplitude with n external partons and l loops. $d\phi_n$ is the phase space measure for n partons. Taken separately, each of the individual contributions $d\sigma_{n+2}^{(0)}$, $d\sigma_{n+1}^{(1)}$ and $d\sigma_n^{(2)}$ gives a divergent contribution. Only the sum of all contributions is finite. However, since one would like to perform the integration numerically, one must first cancel analytically the divergences, before numerical integration over the different phase spaces can be envisaged. This is not a new phenomena at NNLO, it already occurs at NLO. We can consider an observable, whose LO contribution depends on $n+1$ partons. The NLO contribution is given by

$$\langle \mathcal{O} \rangle_{n+1}^{NLO} = \int \mathcal{O}_{n+2}(p_1,\ldots,p_{n+2}) d\sigma_{n+2}^{(0)} + \int \mathcal{O}_{n+1}(p_1,\ldots,p_{n+1}) d\sigma_{n+1}^{(1)}\,. \tag{2.4}$$

To render the two contributions finite, one adds and subtracts a suitable term [15]

$$\begin{aligned} \int \mathcal{O}_{n+2}\, d\sigma_{n+2}^{(0)} + \int \mathcal{O}_{n+2}\, d\sigma_{n+1}^{(1)} = & \int \left(\mathcal{O}_{n+2}\, d\sigma_{n+2}^{(0)} - \mathcal{O}_{n+1} \circ d\alpha_{n+1}^{(0,1)} \right) + \\ & + \int \left(\mathcal{O}_{n+1}\, d\sigma_{n+1}^{(1)} + \mathcal{O}_{n+1} \circ d\alpha_{n+1}^{(0,1)} \right). \end{aligned} \tag{2.5}$$

The approximation $d\alpha_{n+1}^{(0,1)}$ has to fulfill the following requirements: First, $d\alpha_{n+1}^{(0,1)}$ must be a proper approximation of $d\sigma_{n+2}^{(0)}$ such as to have the same pointwise singular behaviour in D dimensions as $d\sigma_{n+2}^{(0)}$ itself. Thus, $d\alpha_{n+1}^{(0,1)}$ acts as a local counterterm for $d\sigma_{n+2}^{(0)}$ and one can safely perform the limit $\varepsilon \to 0$. Secondly, $d\sigma_{n+1}^{(0,1)}$ must be analytically integrable in D dimensions over the one-parton subspace leading to soft and collinear divergences. The integral

$$\int_1 d\alpha_{n+1}^{(0,1)} \tag{2.6}$$

contains then explicit poles in $1/\varepsilon^2$ and $1/\varepsilon$ which exactly cancel the poles from the loop amplitude in $d\sigma_{n+1}^{(1)}$. Note that the observable $\mathcal{O}$ in the approximation term is evaluated with a $n+1$ parton configuration. This allows the integration over the one-parton subspace to be performed analytically, but requires a mapping of four-momenta from the $n+2$ parton configuration to the $n+1$ parton configuration. This mapping has to satisfy momentum conservation and also has to keep all partons massless. In addition it has to have the right behaviour in the singular limits. Several choices for such a mapping exist [15, 37]. The notation $\mathcal{O}_{n+1} \circ d\alpha_{n+1}^{(0,1)}$ is a reminder, that in general the approximation is a sum of terms

$$\mathcal{O}_{n+1} \circ d\alpha_{n+1}^{(0,1)} = \sum \mathcal{O}_{n+1}\, d\alpha_{n+1}^{(0,1)} \tag{2.7}$$

and the mapping used to relate the $n+2$ parton configuration to a $n+1$ parton configuration differs in general for each summand.

In a similar way, I rewrite the NNLO contribution as

$$\begin{aligned}\langle \mathcal{O} \rangle_n^{NNLO} = & \int \left(\mathcal{O}_{n+2}\, d\sigma_{n+2}^{(0)} - \mathcal{O}_{n+1} \circ d\alpha_{n+1}^{(0,1)} - \mathcal{O}_n \circ d\alpha_n^{(0,2)} \right) + \\ & + \int \left(\mathcal{O}_{n+1}\, d\sigma_{n+1}^{(1)} + \mathcal{O}_{n+1} \circ d\alpha_{n+1}^{(0,1)} - \mathcal{O}_n \circ d\alpha_n^{(1,1)} \right) + \\ & + \int \left(\mathcal{O}_n\, d\sigma_n^{(2)} + \mathcal{O}_n \circ d\alpha_n^{(0,2)} + \mathcal{O}_n \circ d\alpha_n^{(1,1)} \right).\end{aligned} \tag{2.8}$$

Here $d\alpha_n^{(0,2)}$ acts as a local counterterm for double unresolved configurations to $d\sigma_{n+2}^{(0)} - d\alpha_{n+1}^{(0,1)}$ in D dimensions. Furthermore, it is integrable over a two-parton subspace. $d\alpha_n^{(1,1)}$ is a local counterterm to $d\sigma_{n+1}^{(1)} + d\alpha_{n+1}^{(0,1)}$ in D dimensions, whenever one of the $(n+1)$ partons becomes unresolved. It is integrable over a one-parton subspace. This setup extends to higher orders and in general, $d\alpha_n^{(l,k)}$ is an approximation to

$$d\sigma_{n+k}^{(l)} + d\alpha_{n+k}^{(0,l)} + d\alpha_{n+k}^{(1,l-1)} + \ldots + d\alpha_{n+k}^{(l-1,1)} - d\alpha_{n+k-1}^{(l,1)} - d\alpha_{n+k-2}^{(l,2)} - \ldots - d\alpha_{n+1}^{(l,k-1)}, \tag{2.9}$$

e.g. a contribution with at most $n+k$ partons and at most l loops in the case when k partons become unresolved.

The task at NNLO is to give the explicit forms of $d\alpha_n^{(0,2)}$ (and of $d\alpha_n^{(0,1)}$ and $d\alpha_n^{(1,1)}$) and to integrate these approximations over a two-parton phase space (or a one-parton phase space in the case of $d\alpha_n^{(0,1)}$ and $d\alpha_n^{(1,1)}$).

To illustrate the cancellations for double unresolved contributions, I consider a simple toy model, where the double real emission contribution to a 1-jet observable is given by

$$\mathcal{O}_3\, d\sigma_3^{(0)} = \theta_3(x_1, x_2, x_3) F_3(x_1, x_2, x_3)\,. \tag{2.10}$$

The singular behaviour of the functions F_3 is assumed to be

$$\lim_{x_1\to 0} F_3(x_1,x_2,x_3) = \frac{1}{x_1}F_2(x_2,x_3)\,,$$
$$\lim_{x_1\to 0} F_2(x_1,x_2) = \frac{1}{x_1}F_1(x_2)\,. \tag{2.11}$$

The function θ_3, defining the observable satisfies

$$\lim_{x_1\to 0} \theta_3(x_1,x_2,x_3) = \theta_2(x_2,x_3)\,,$$
$$\lim_{x_1\to 0} \theta_2(x_1,x_2) = \theta_1(x_2)\,. \tag{2.12}$$

Similar relations are assumed to hold for the singular limits in the other variables. The NLO subtraction term is given by

$$\mathcal{O}_2 \circ d\alpha_2^{(0,1)} = \frac{1}{x_1}F_2(x_2,x_3)\theta_2(x_2,x_3) + \frac{1}{x_2}F_2(x_1,x_3)\theta_2(x_1,x_3) + $$
$$+\frac{1}{x_3}F_2(x_1,x_2)\theta_2(x_1,x_2)\,. \tag{2.13}$$

Let us first consider the cancellation of single unresolved contributions to a 2-jet observable. I consider the $x_1 \to 0$ limit. Here, $\mathcal{O}_3\ d\sigma_3^{(0)} - \mathcal{O}_2 \circ d\alpha_2^{(0,1)}$ consists of two contributions: The first part

$$\lim_{x_1\to 0}\left(F_3(x_1,x_2,x_3)\theta_3(x_1,x_2,x_3) - \frac{1}{x_1}F_2(x_2,x_3)\theta_2(x_2,x_3)\right) \tag{2.14}$$

is finite by construction. The second part is given by

$$\lim_{x_1\to 0}\left(-\frac{1}{x_2}F_2(x_1,x_3)\theta_2(x_1,x_3) - \frac{1}{x_3}F_2(x_1,x_2)\theta_2(x_1,x_2)\right) =$$
$$= -\frac{1}{x_1x_2}F_1(x_3)\theta_1(x_3) - \frac{1}{x_1x_3}F_1(x_2)\theta_1(x_2)\,. \tag{2.15}$$

This part is proportional to $\theta_1(x_2)$ or $\theta_1(x_3)$ and therefore does not contribute to a 2-jet observable (θ_1 is zero for a two-jet observable). However, this argument no longer holds if we move to NNLO and consider double unresolved contributions to a 1-jet observable. Here, θ_1 does not vanish and the r.h.s of eq. (2.15) gives a divergent contribution already in the single unresolved case $x_1 \to 0$, and $x_2, x_3 \neq 0$. At NNLO these singular pieces have to be taken properly into account and subtracted out. To summarize, the major complication for the subtraction method is not to find subtraction terms which match the matrix element in all double unresolved limits, but to find subtraction terms, which match in all double unresolved limits **and** all single unresolved limits. For the toy example the NNLO subtraction term is given by

$$\mathcal{O}_1 \circ d\alpha_2^{(0,2)} = -\frac{1}{x_1x_2}F_1(x_3)\theta_1(x_3) - \frac{1}{x_1x_3}F_1(x_2)\theta_1(x_2) - \frac{1}{x_2x_3}F_1(x_1)\theta_1(x_1)\,. \tag{2.16}$$

Note that in this toy example the NNLO subtraction term can be obtained solely from the singular behaviour of the NLO subtraction term. In general however, the limits do not commute, e.g. taking the singular limit of the single unresolved approximation does not equal the double unresolved approximation of the matrix element.

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3. An example

To make the method for the double unresolved contributions as transparent as possible, I consider as a simple example the leading N_c-contribution to the tree-level amplitude $e^+e^- \to qgg\bar{q}$. This example is complicated enough to expose all essential features.

3.1 Double real emission

The singular behaviour of the kinematical structure of QCD amplitudes is entangled with colour factors. For a clear understanding of the factorization properties in soft and collinear limits it is desirable to disentangle the colour factors from the rest. This can be done either by introducing colour charge operators [15] or by decomposing the full amplitude into partial and primitive amplitudes. In this paper I follow the second approach, since this method is closer related to the way several one-loop amplitudes were calculated. Throughout this paper I use the following normalization for the colour matrices:

$$\mathrm{Tr}\, T^a T^b = \frac{1}{2}\delta^{ab}\,. \tag{3.1}$$

Tree-level amplitudes are decomposed into colour factors and partial amplitudes. Each partial amplitude is gauge-invariant and has usually a simpler structure. For example, the colour decompostion of the the tree amplitudes for $e^+e^- \to q_1 g_2 g_3 \bar{q}_4$ amplitude reads

$$\mathcal{A}_4^{(0)}(q_1,g_2,g_3,\bar{q}_4) = e^2 g^2 c_0 \left[\left(T^2 T^3\right)_{14} A_4^{(0)}(q_1,g_2,g_3,\bar{q}_4) + \left(T^3 T^2\right)_{14} A_4^{(0)}(q_1,g_3,g_2,\bar{q}_4)\right]. \tag{3.2}$$

Here, c_0 denotes a factor from the electro-magnetic coupling:

$$\begin{aligned} c_0 &= -Q^q + v^e v^q \mathcal{P}_Z(s)\,, \\ \mathcal{P}_Z(s) &= \frac{s}{s - M_Z^2 + i\Gamma_Z M_Z}\,. \end{aligned} \tag{3.3}$$

The electron-positron pair can either annihilate into a photon or a Z-boson. The first term in the expression for c_0 corresponds to an intermediate photon, whereas the last term corresponds to a Z-boson. The left- and right-handed couplings of the Z-boson to fermions are given by

$$v_L = \frac{I_3 - Q\sin^2\theta_W}{\sin\theta_W \cos\theta_W}\,, \qquad v_R = \frac{-Q\sin\theta_W}{\cos\theta_W}\,, \tag{3.4}$$

where Q and I_3 are the charge and the third component of the weak isospin of the fermion. The short-hand notation for the colour-matrices corresponds to $T^2_{14} = T^{a_g}_{i_q j_{\bar{q}}}$. Figure 2 shows the diagrams contributing to the colour-ordered partial amplitude $A_4^{(0)}(q_1,g_2,g_3,\bar{q}_4)$. Squaring the full amplitude $\mathcal{A}_4^{(0)}(q_1,g_2,g_3,\bar{q}_4)$ one obtains

$$\begin{aligned} \mathcal{A}_4^{(0)\,*} \mathcal{A}_4^{(0)} &= (4\pi\alpha)^2 (4\pi\alpha_s)^2 \left|c_0\right|^2 \times \qquad (3.5) \\ &\times \frac{1}{4}\left(N_c^2 - 1\right) N_c \left[\left|A_4^{(0)}(q_1,g_2,g_3,\bar{q}_4)\right|^2 + \left|A_4^{(0)}(q_1,g_3,g_2,\bar{q}_4)\right|^2 + O\left(\frac{1}{N_c^2}\right)\right]. \end{aligned}$$

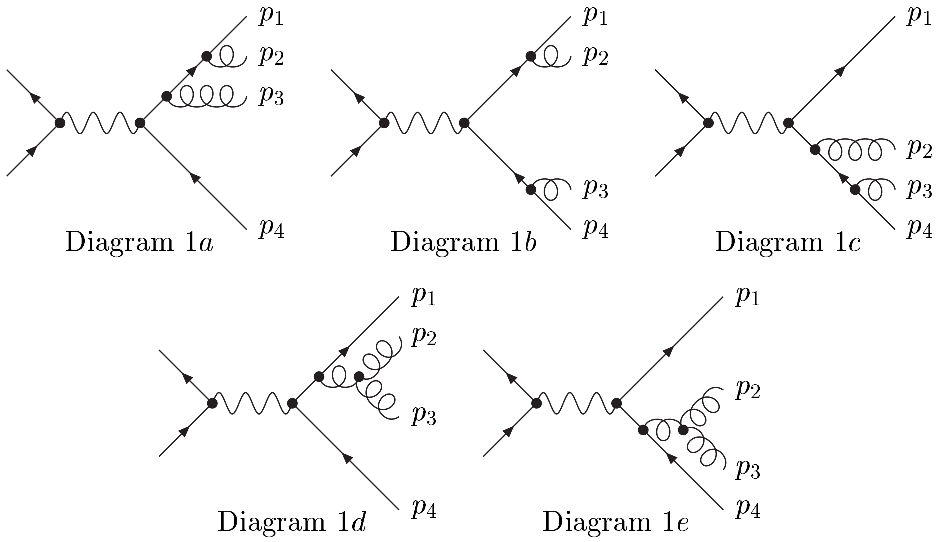

Figure 2: Diagrams contributing to the colour-ordered partial amplitude $A_4^{(0)}(q_1, g_2, g_3, \bar{q}_4)$.

Note that interference terms between $A_4^{(0)}(q_1, g_2, g_3, \bar{q}_4)$ and $A_4^{(0)}(q_1, g_3, g_2, \bar{q}_4)$ are subleading in N_c. In four dimensions compact expressions for the partial amplitudes are obtained by calculating helicity amplitudes. For the partial amplitude $A_4^{(0)}(q_1, g_2, g_3, \bar{q}_4)$ there are only three independent helicity configurations, which can be taken to be

$$q_1^+, g_2^+, g_3^+, \bar{q}_4^-, p_5^-, p_6^+ \,, \qquad q_1^+, g_2^+, g_3^-, \bar{q}_4^-, p_5^-, p_6^+ \,, \qquad q_1^+, g_2^-, g_3^+, \bar{q}_4^-, p_5^-, p_6^+ \,. \tag{3.6}$$

The helicity amplitudes are expressed in term of spinorproducts and read [38]:

$$\begin{aligned}
A_4^{(0)}(q_1^+, g_2^+, g_3^+, \bar{q}_4^-, p_5^-, p_6^+) &= -4\,i\,\frac{\langle 4\,5\rangle^2}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 5\,6\rangle}\,, \qquad (3.7)\\
A_4^{(0)}(q_1^+, g_2^+, g_3^-, \bar{q}_4^-, p_5^-, p_6^+) &= 4\,i\Bigg[\frac{\langle 3\,1\rangle\,[1\,2]\,\langle 4\,5\rangle\,\langle 3|(1{+}2)|6\rangle}{\langle 1\,2\rangle\,s_{23}s_{123}s_{56}} - \frac{\langle 3\,4\rangle\,[4\,2]\,[1\,6]\,\langle 5|(3{+}4)|2\rangle}{[3\,4]\,s_{23}s_{234}s_{56}} - \\
&\qquad - \frac{\langle 5|(3+4)|2\rangle\,\langle 3|(1+2)|6\rangle}{\langle 1\,2\rangle\,[3\,4]\,s_{23}s_{56}}\Bigg]\,,\\
A_4^{(0)}(q_1^+, g_2^-, g_3^+, \bar{q}_4^-, p_5^-, p_6^+) &= 4\,i\Bigg[-\frac{[1\,3]^2\,\langle 4\,5\rangle\,\langle 2|(1+3)|6\rangle}{[1\,2]\,s_{23}s_{123}s_{56}} + \frac{\langle 2\,4\rangle^2\,[1\,6]\,\langle 5|(2+4)|3\rangle}{\langle 3\,4\rangle\,s_{23}s_{234}s_{56}} + \\
&\qquad + \frac{[1\,3]\,\langle 2\,4\rangle\,[1\,6]\,\langle 4\,5\rangle}{[1\,2]\,\langle 3\,4\rangle\,s_{23}s_{56}}\Bigg]\,.
\end{aligned}$$

The notation of spinor products follows [38]. For the singular limits it is sufficient to know that spinor products scale as the square root of invariants $s_{ij} = (p_i + p_j)^2$:

$$|\langle i\,j\rangle| = |[i\,j]| = \sqrt{s_{ij}}\,. \tag{3.8}$$

For the last two helicity amplitudes there are alternate forms, which have a clearer singular structure in the $p_2||p_3$ limit, at the expense of a more obscure behaviour in the triple collinear limits $p_1||p_2||p_3$ and $p_2||p_3||p_4$:

$$A_4^{(0)}(q_1^+, g_2^+, g_3^-, \bar{q}_4^-, p_5^-, p_6^+) = 4\,i\left[\frac{\langle 5\,4\rangle\,[4\,2]\,[1\,2]\,\langle 5|(3+4)|2\rangle}{[2\,3]\,[3\,4]\,s_{123}s_{234}\,\langle 5\,6\rangle} + \frac{\langle 3\,1\rangle\,[1\,6]\,\langle 3\,4\rangle\,\langle 3|(1+2)|6\rangle}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,s_{123}s_{234}\,[5\,6]} - \right.$$
$$\left. - \frac{\langle 3|(1+2)|6\rangle\,\langle 5|(3+4)|2\rangle}{\langle 1\,2\rangle\,[3\,4]\,s_{123}s_{234}}\right],$$
$$A_4^{(0)}(q_1^+, g_2^-, g_3^+, \bar{q}_4^-, p_5^-, p_6^+) = 4\,i\left[\frac{[1\,3]^2\,\langle 4\,5\rangle\,\langle 5|(2+4)|3\rangle}{[1\,2]\,[2\,3]\,s_{123}s_{234}\,\langle 5\,6\rangle} - \frac{\langle 2\,4\rangle^2\,[1\,6]\,\langle 2|(1+3)|6\rangle}{\langle 2\,3\rangle\,\langle 3\,4\rangle\,s_{123}s_{234}\,[5\,6]} + \right.$$
$$\left. + \frac{[1\,3]\,\langle 2\,4\rangle\,[1\,6]\,\langle 4\,5\rangle}{[1\,2]\,\langle 3\,4\rangle\,s_{123}s_{234}}\right]. \tag{3.9}$$

As can be seen from these explicit formulae there are simultaneous singularities in single collinear limits $s_{ij} \to 0$ and triple collinear limits $s_{ijk} \to 0$. These are nested singularities. It should be noted that the helicity amplitudes are sufficient for the derivation of the subtraction terms when working in a four-dimensional regularization scheme [39]–[41]. In conventional dimensional regularization [42] the squared matrix elements has $O(\varepsilon)$-terms, which also have to be taken into account.

3.2 One-loop amplitudes with one unresolved parton

For a NNLO analysis of $e^+e^- \to$ 2 jets one needs also the amplitude for $e^+e^- \to qg\bar{q}$ up to one loop. The colour decomposition of the tree amplitudes and the one-loop amplitudes are

$$\mathcal{A}_3^{(0)}(q_1, g_2, \bar{q}_3) = e^2 g c_0 \left(T^2\right)_{13} A_3^{(0)}(q_1, g_2, \bar{q}_3),$$
$$\mathcal{A}_3^{(1)}(q_1, g_2, \bar{q}_3) = e^2 g c_0 \left(T^2\right)_{13} \left(\frac{\alpha_s}{2\pi}\right) A_{3,\text{partial}}^{(1)}(q_1, g_2, \bar{q}_3). \tag{3.10}$$

For both the tree amplitude and the one-loop amplitude there is only one partial amplitude. However, to obtain a clear factorization in singular kinematical limits, it is convenient to decompose the one-loop partial amplitude further into primitive amplitudes [31]. Primitive amplitudes can be defined as the sum of all Feynman diagrams with a fixed cyclic ordering of the QCD partons and a definite routing of the external fermion lines through the diagram. For the case at hand we have

$$A_{3,\text{partial}}^{(1)}(q_1, g_2, \bar{q}_3) = \frac{1}{2} N_c A_3^{(1),L,[1]}(q_1, g_2, \bar{q}_3) - \frac{1}{2N_c} A_3^{(1),R,[1]}(q_1, g_2, \bar{q}_3) +$$
$$+ \frac{1}{2} N_f A_3^{(1),L,[1/2]}(q_1, g_2, \bar{q}_3). \tag{3.11}$$

The superscript L or R indicates whether the external fermion line turns left or right when entering the loop, while the superscripts [1] and [1/2] indicate the spin of the particle circulating in the loop. Examples of one-loop diagrams corresponding to different primitive amplitudes are shown in figure 3. Since we are here only interested in the leading-colour contribution, only the primitive amplitude $A_3^{(1),L,[1]}(q_1, g_2, \bar{q}_3)$ is relevant. In later sections I will drop the superscript "$L, [1]$" for the leading-colour primitive amplitude. There is only one independent helicity configuration, which we can take to be

$$q_1^+, g_2^+, \bar{q}_3^-, p_4^-, p_5^+. \tag{3.12}$$

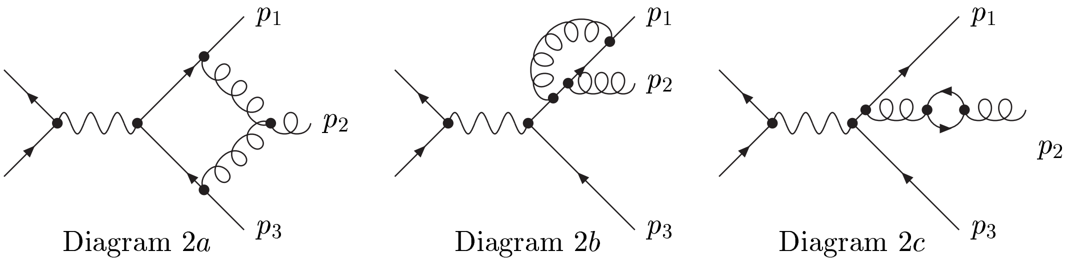

Figure 3: Examples of one-loop diagrams, contributing to different primitive amplitudes: Diagram 2a contributes to $A_3^{(1),L,[1]}$, diagram 2b contributes to $A_3^{(1),R,[1]}$ and diagram 2c contributes to $A_3^{(1),L,[1/2]}$.

The helicity amplitudes are expressed in terms of spinorproducts and read:

$$\begin{aligned}
A_3^{(0)}(q_1^+, g_2^+, \bar{q}_3^-, p_4^-, p_5^+) &= 2\sqrt{2}i\frac{\langle 34\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 54\rangle}\,, \\
A_3^{(1),L,[1]}(q_1^+, g_2^+, \bar{q}_3^-, p_4^-, p_5^+) &= V_3^{(1)} A_3^{(0)}(q_1^+, g_2^+, \bar{q}_3^-, p_4^-, p_5^+) + \\
&\quad +2\sqrt{2}i\frac{\langle 13\rangle[15]\langle 34\rangle}{\langle 12\rangle\langle 23\rangle s_{45}} F_0 + 2\sqrt{2}i\frac{\langle 13\rangle^2[15]^2\langle 45\rangle}{\langle 12\rangle\langle 23\rangle s_{45}^2} F_1\,. \qquad (3.13)
\end{aligned}$$

The renormalized coefficients $V_3^{(1)}$, F_0, and F_1 are given in the HV scheme by

$$\begin{aligned}
V_3^{(1)} &= -\frac{2}{\varepsilon^2} + \left(\ln(x_1) + \ln(x_2) - \frac{10}{3}\right)\frac{1}{\varepsilon} - R(x_1,x_2) - \frac{1}{2}\ln^2(x_1) - \frac{1}{2}\ln^2(x_2) + \frac{7}{6}\pi^2 + \\
&\quad + \frac{3}{2}\ln(x_2) - \frac{7}{2} + O(\varepsilon) + \text{imaginary parts}\,, \\
F_0 &= \frac{\ln(x_2)}{(1-x_2)} + O(\varepsilon)\,, \\
F_1 &= \frac{1}{2}\frac{(1-x_2) + \ln(x_2)}{(1-x_2)^2} + O(\varepsilon)\,. \qquad (3.14)
\end{aligned}$$

Here, $x_1 = s_{12}/s_{123}$, $x_2 = s_{23}/s_{123}$ and the symmetric function $R(x_1,x_2)$ is defined by

$$R(x_1,x_2) = \left(\frac{1}{2}\ln(x_1)\ln(x_2) - \ln(x_1)\ln(1-x_1) + \frac{1}{2}\zeta_2 - \mathrm{Li}_2(x_1)\right) + (x_1 \leftrightarrow x_2)\,. \qquad (3.15)$$

The amplitude in the FDH scheme is obtained by replacing $V_3^{(1)}$ by

$$V_{3,FDH}^{(1)} = V_3^{(1)} + \frac{1}{2}\,. \qquad (3.16)$$

3.3 The two-loop amplitude

For a NNLO calculation one also needs the amplitudes for $e^+e^- \to q\bar{q}$ up to two loops. The colour decompositions of the tree amplitudes, the one-loop amplitudes and the two-loop amplitudes are

$$\begin{aligned}
\mathcal{A}_2^{(0)}(q_1,\bar{q}_2) &= e^2 c_0 \delta_{12} A_2^{(0)}(q_1,\bar{q}_2)\,, \\
\mathcal{A}_2^{(1)}(q_1,\bar{q}_2) &= e^2 c_0 \delta_{12}\left(\frac{\alpha_s}{2\pi}\right) A_{2,\text{partial}}^{(1)}(q_1,\bar{q}_2)\,, \\
\mathcal{A}_2^{(2)}(q_1,\bar{q}_2) &= e^2 c_0 \delta_{12}\left(\frac{\alpha_s}{2\pi}\right)^2 A_{2,\text{partial}}^{(2)}(q_1,\bar{q}_2)\,. \qquad (3.17)
\end{aligned}$$

There is only one independent helicity configuration, which we can take to be

$$q_1^+, \bar{q}_2^-, p_3^-, p_4^+ . \tag{3.18}$$

The helicity amplitude expressed in terms of spinorproducts reads:

$$A_2^{(0)}\left(q_1^+, \bar{q}_2^-, p_3^-, p_4^+\right) = 2i\frac{[14]\langle 32\rangle}{s_{12}} . \tag{3.19}$$

The leading colour contributions to $A_2^{(1)}$ and $A_2^{(2)}$ are:

$$\begin{aligned} A^{(1)}_{2,\text{partial}}\left(q_1^+, \bar{q}_2^-, p_3^-, p_4^+\right) &= \frac{1}{2}N_c V_2^{(1)} A_2^{(0)}\left(q_1^+, \bar{q}_2^-, p_3^-, p_4^+\right) + O\left(\frac{1}{N_c^2}\right), \\ A^{(2)}_{2,\text{partial}}\left(q_1^+, \bar{q}_2^-, p_3^-, p_4^+\right) &= \frac{1}{4}\left(N_c^2-1\right) V_2^{(2)} A_2^{(0)}\left(q_1^+, \bar{q}_2^-, p_3^-, p_4^+\right) + O\left(\frac{1}{N_c}\right). \end{aligned} \tag{3.20}$$

The leading-colour primitive amplitudes are therefore

$$\begin{aligned} A_2^{(1)} &= V_2^{(1)} A_2^{(0)}, \\ A_2^{(2)} &= V_2^{(2)} A_2^{(0)}. \end{aligned} \tag{3.21}$$

The renormalized coefficients $V_2^{(1)}$ and $V_2^{(2)}$ are given in the HV scheme by [43]–[45]

$$\begin{aligned} V_2^{(1)} &= -\frac{1}{\varepsilon^2} - \frac{3}{2\varepsilon} - 4 + \frac{7}{12}\pi^2 + \left(-8 + \frac{7}{8}\pi^2 + \frac{7}{3}\zeta_3\right)\varepsilon + \left(-16 + \frac{7}{3}\pi^2 + \frac{7}{2}\zeta_3 - \frac{73}{1440}\pi^4\right)\varepsilon^2 \\ &\quad + i\pi\left[-\frac{1}{\varepsilon} - \frac{3}{2} + \left(-4 + \frac{\pi^2}{4}\right)\varepsilon + \left(-8 + \frac{3}{8}\pi^2 + \frac{7}{3}\zeta_3\right)\varepsilon^2\right] + O(\varepsilon^3), \\ V_2^{(2)} &= \frac{1}{2\varepsilon^4} + \frac{17}{4\varepsilon^3} + \left(\frac{433}{72} - \pi^2\right)\frac{1}{\varepsilon^2} + \left(\frac{4045}{432} - \frac{83}{24}\pi^2 + \frac{7}{6}\zeta_3\right)\frac{1}{\varepsilon} - \frac{9083}{2592} - \frac{2153}{432}\pi^2 + \\ &\quad + \frac{26}{9}\zeta_3 + \frac{263}{720}\pi^4 + O(\varepsilon) + \text{imaginary parts}. \end{aligned} \tag{3.22}$$

Note that for a NNLO calculation of $e^+e^- \to$ 2 jets one needs the real part of $V_2^{(1)}$ to order $O(\varepsilon^2)$ and the imaginary part of $V_2^{(1)}$ to order $O(\varepsilon)$.

3.4 Ultraviolet renormalization

The amplitudes in the examples above are the renormalized ones, i.e. the ultraviolet subtraction has been performed. To obtain the renormalized amplitudes in the $\overline{\text{MS}}$ scheme from the bare ones, one replaces the bare coupling α_0 with the renormalized coupling $\alpha_s(\mu^2)$ evaluated at the renormalization scale μ^2:

$$\alpha_0 = \alpha_s S_\varepsilon^{-1}\mu^{2\varepsilon}\left[1 - \frac{\beta_0}{\varepsilon}\left(\frac{\alpha_s}{2\pi}\right) + \left(\frac{\beta_0^2}{\varepsilon^2} - \frac{\beta_1}{2\varepsilon}\right)\left(\frac{\alpha_s}{2\pi}\right)^2 + \mathcal{O}\left(\alpha_s^3\right)\right], \tag{3.23}$$

where

$$S_\varepsilon = (4\pi)^\varepsilon e^{-\varepsilon\gamma_E}, \tag{3.24}$$

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is the typical phase-space volume factor in $D = 4 - 2\varepsilon$ dimensions, γ_E is Euler's constant, and β_0 and β_1 are the first two coeffcients of the QCD β-function:

$$\beta_0 = \frac{11}{6} C_A - \frac{2}{3} T_R N_f \,, \qquad \beta_1 = \frac{17}{6} C_A^2 - \frac{5}{3} C_A T_R N_f - C_F T_R N_f \,, \tag{3.25}$$

with the color factors

$$C_A = N \,, \qquad C_F = \frac{N^2-1}{2N} \,, \qquad T_R = \frac{1}{2} \,. \tag{3.26}$$

Let the expansion in the strong coupling of the unrenormalized amplitude be

$$\begin{aligned} \mathcal{A}_{n,bare} = {}& (4\pi\alpha)\,(4\pi\alpha_0)^{\frac{n-2}{2}}\, c_0\, \mathcal{C} \left[A_{n,bare}^{(0)} + \left(\frac{\alpha_0}{2\pi}\right) A_{n,bare}^{(1)} + \left(\frac{\alpha_0}{2\pi}\right)^2 A_{n,bare}^{(2)} + O(\alpha_s^3) + \right] + \\ & + \text{subleading colour structures.} \end{aligned} \tag{3.27}$$

where $\mathcal{C}$ is a colour factor. Then, the renormalized two-loop amplitude can be expessed as

$$\begin{aligned} \mathcal{A}_{n,ren} = {}& (4\pi\alpha)\,(4\pi\alpha_s)^{\frac{n-2}{2}}\, c_0\, \mathcal{C}\, \left(S_\varepsilon^{-1}\mu^{2\varepsilon}\right)^{\frac{n-2}{2}} \times \\ & \times \left[A_{n,ren}^{(0)} + \left(\frac{\alpha_s}{2\pi}\right) A_{n,ren}^{(1)} + \left(\frac{\alpha_s}{2\pi}\right)^2 A_{n,ren}^{(2)} + O(\alpha_s^3) \right] + \text{subleading colour structures}\,. \end{aligned} \tag{3.28}$$

The relations between the renormalized and the bare amplitudes are given by

$$\begin{aligned} A_{n,ren}^{(0)} &= A_{n,bare}^{(0)} \,, \\ A_{n,ren}^{(1)} &= S_\varepsilon^{-1}\mu^{2\varepsilon} A_{n,bare}^{(1)} - \frac{(n-2)}{2}\frac{\beta_0}{\varepsilon} A_{n,bare}^{(0)} \,, \\ A_{n,ren}^{(2)} &= S_\varepsilon^{-2}\mu^{4\varepsilon} A_{n,bare}^{(2)} - \frac{n}{2}\frac{\beta_0}{\varepsilon} S_\varepsilon^{-1}\mu^{2\varepsilon} A_{n,bare}^{(1)} + \frac{(n-2)}{2}\left(n\frac{\beta_0^2}{4\varepsilon^2} - \frac{\beta_1}{2\varepsilon} \right) A_{n,bare}^{(0)} \,. \end{aligned} \tag{3.29}$$

Below I will discuss the factorization of one-loop amplitudes in singular (soft and collinear) limits. The general structure is

$$A_n^{(1)} = \text{Sing}^{(0)} \cdot A_{n-1}^{(1)} + \text{Sing}^{(1)} \cdot A_{n-1}^{(0)}, \tag{3.30}$$

where the function Sing corresponds either to the eikonal factor (soft limit) or a splitting function (collinear limit). Eq. (3.30) holds both for bare and renormalized quantitites. The relation between the renormalized and the bare expression for the singular function is

$$\text{Sing}_{ren}^{(1)} = S_\varepsilon^{-1}\mu^{2\varepsilon}\, \text{Sing}_{bare}^{(1)} - \frac{\beta_0}{2\varepsilon}\, \text{Sing}_{bare}^{(0)} \,. \tag{3.31}$$

4. Singular limits

In this section I review the behaviour of amplitudes in singular limits, e.g. when two or three partons become degenerate. The knowledge of the singular limits serves as a starting point for the construction of the subtraction terms. Most of the material in this section is summarized from the literature [19]–[33], although some results, like the triple collinear splitting function for colour-ordered amplitudes, are new. When considering singular limits it is convenient to work in a physical gauge. In this case, only diagrams where the splitting occurs at external lines contribute [46, 24]. In this paper I use the axial gauge $n \cdot A = 0$. In this gauge the gluon propagator reads

$$\begin{aligned}\frac{i}{k^2} d^{\mu\nu}(k,n) &= \frac{i}{k^2}\left(-g^{\mu\nu} + \frac{k^\mu n^\nu + n^\mu k^\nu}{k \cdot n} - n^2 \frac{k^\mu k^\nu}{(k\cdot n)^2}\right) \\ &= -\frac{i}{k^2}\left(g^{\mu\rho} - \frac{k^\mu n^\rho}{k\cdot n}\right)\left(g_\rho{}^\nu - \frac{n_\rho k^\nu}{k\cdot n}\right). \end{aligned} \tag{4.1}$$

4.1 Soft gluons

If a single gluon becomes soft, the partial tree amplitude factorizes according to

$$A_n^{(0)}(p_1, p_2, p_3, \ldots) = \mathrm{Eik}^{(0,1)}(p_1, p_2, p_3) A_{n-1}^{(0)}(p_1, p_3, \ldots)\,, \tag{4.2}$$

where the eikonal factor is given by

$$\mathrm{Eik}^{(0,1)}(p_1,p_2,p_3) = 4 \frac{p_1^\rho F_{\rho\mu\sigma}(p_2) p_3^\sigma}{s_{12} s_{23}} \varepsilon^\mu(p_2) \tag{4.3}$$

and $F_{\rho\mu\sigma}(p)$ is defined by

$$F^{\rho\mu\sigma}(p) = g^{\rho\mu} p^\sigma - p^\rho g^{\mu\sigma}\,. \tag{4.4}$$

The square of the eikonal factor is given by

$$\left|\mathrm{Eik}^{(0,1)}(p_1,p_2,p_3)\right|^2 = 4 \frac{s_{13}}{s_{12}s_{23}}\,. \tag{4.5}$$

In a similar way the partial tree amplitude factorizes, when two gluons become soft [20]

$$A_n^{(0)}(p_1, p_2, p_3, p_4, \ldots) = \mathrm{Eik}^{(0,2)}(p_1,p_2,p_3,p_4) A_{n-2}^{(0)}(p_1, p_4, \ldots), \tag{4.6}$$

with

$$\begin{aligned}\mathrm{Eik}^{(0,2)}(p_1,p_2,p_3,p_4) = 8\Bigg[&\frac{p_1^\rho F_{\rho\mu\sigma}(p_2) F_{\sigma\nu\tau}(p_3) p_4^\tau}{s_{12}s_{23}s_{34}} - \frac{p_1^\rho F_{\rho\mu\sigma}(p_2) F_{\sigma\nu\tau}(p_3) p_1^\tau}{s_{12}s_{23}(s_{12}+s_{13})} - \\ &- \frac{p_4^\rho F_{\rho\mu\sigma}(p_2) F_{\sigma\nu\tau}(p_3) p_4^\tau}{s_{23}s_{34}(s_{24}+s_{34})}\Bigg] \varepsilon^\mu(p_2)\varepsilon^\nu(p_3)\,. \end{aligned} \tag{4.7}$$

The square of the double soft eikonal factor is given by

$$\begin{aligned}\left|\mathrm{Eik}^{(0,2)}(p_1,p_2,p_3,p_4)\right|^2 = 8\Bigg[&(1-\rho\varepsilon)\frac{(s_{123}s_{34} + s_{12}s_{234} - s_{123}s_{234})^2}{s_{123}^2 s_{23}^2 s_{234}^2} + \frac{s_{14}^2}{s_{12}s_{34}s_{123}s_{234}} + \\ &+ \frac{s_{14}}{s_{23}}\left(\frac{1}{s_{12}s_{34}} + \frac{1}{s_{12}s_{234}} + \frac{1}{s_{123}s_{34}} - \frac{4}{s_{123}s_{234}}\right)\Bigg]. \end{aligned} \tag{4.8}$$

I introduced the parameter ρ, which specifies the variant of dimensional regularization: $\rho = 1$ for the CDR/HV schemes and $\rho = 0$ for the FD scheme. In the soft-gluon limit, a one-loop primitive amplitude factorizes according to

$$\begin{aligned}A_n^{(1)}(p_1,p_2,p_3,\ldots) = &\ \mathrm{Eik}^{(0,1)}(p_1,p_2,p_3) \cdot A_{n-1}^{(1)}(p_1,p_3,\ldots) + \\ &+ \mathrm{Eik}^{(1,1)}(p_1,p_2,p_3) \cdot A_{n-1}^{(0)}(p_1,p_3,\ldots)\,, \end{aligned} \tag{4.9}$$

For the example considered here, the relevant one-loop eikonal function is given by [31]

$$\mathrm{Eik}^{(1,1)}(p_1,p_2,p_3) = \left[-\frac{S_\varepsilon^{-1} c_\Gamma}{\varepsilon^2} \Gamma(1+\varepsilon)\Gamma(1-\varepsilon) \left(\frac{\mu^2(-s_{13})}{(-s_{12})(-s_{23})} \right)^\varepsilon - \frac{11}{6\varepsilon} \right] \mathrm{Eik}^{(0,1)}(p_1,p_2,p_3). \tag{4.10}$$

Here,

$$c_\Gamma = (4\pi)^\varepsilon \frac{\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)}. \tag{4.11}$$

As in the tree-level case with one soft gluon, the eikonal factor is independent of the variant of the dimensional regularization scheme used.

4.2 Collinear particles

In the collinear limit tree amplitudes factorize according to

$$A_n^{(0)}(\ldots, p_i, p_j, \ldots) = \sum_\lambda \mathrm{Split}^{(0,1)}(p_i, p_j)\, A_{n-1}^{(0)}(\ldots, p, \ldots). \tag{4.12}$$

where p_i and p_j are the momenta of two adjacent legs and the sum is over all polarizations. In the collinear limit we parametrize the momenta of the partons i and j as [15]

$$\begin{aligned} p_i &= zp + k_\perp - \frac{k_\perp^2}{z} \frac{n}{2pn}, \\ p_j &= (1-z)p - k_\perp - \frac{k_\perp^2}{1-z} \frac{n}{2pn}. \end{aligned} \tag{4.13}$$

Here n is a massless four-vector and the transverse component $k_\perp$ satisfies $2pk_\perp = 2nk_\perp = 0$. The collinear limits occurs for $k_\perp^2 \to 0$. The splitting amplitudes $\mathrm{Split}^{(0,1)}$ are universal, they depend only on the two momenta becoming collinear, and not upon the specific amplitude under consideration. The splitting functions $\mathrm{Split}^{(0,1)}$ are given by

$$\begin{aligned} \mathrm{Split}^{(0,1)}_{q\to qg} &= \frac{1}{s_{ij}} \bar{u}(p_i) \not\varepsilon(p_j) u(p), \\ \mathrm{Split}^{(0,1)}_{g\to gg} &= \frac{2}{s_{ij}} \left[\varepsilon(p_i)\cdot\varepsilon(p_j)\; p_i\cdot\varepsilon(p) + \varepsilon(p_j)\cdot\varepsilon(p)\; p_j\cdot\varepsilon(p_i) - \varepsilon(p_i)\cdot\varepsilon(p)\; p_i\cdot\varepsilon(p_j) \right], \\ \mathrm{Split}^{(0,1)}_{g\to q\bar{q}} &= \frac{1}{s_{ij}} \bar{u}(p_i) \not\varepsilon(p) u(p_j). \end{aligned} \tag{4.14}$$

Due to the sum over spins in eq. (4.12), spin correlations are retained in the collinear limit of squared tree amplitudes. With

$$\begin{aligned} P^{(0,1)} &= \sum_{\lambda,\lambda'} u(p)\, \mathrm{Split}^{(0,1)\,*}\, \mathrm{Split}^{(0,1)}\, \bar{u}(p) \quad \text{for quarks}, \\ P^{(0,1)} &= \sum_{\lambda,\lambda'} \varepsilon^\mu(p)^*\, \mathrm{Split}^{(0,1)\,*}\, \mathrm{Split}^{(0,1)}\, \varepsilon^\nu(p) \quad \text{for gluons}, \end{aligned} \tag{4.15}$$

and the paramterization eq. (4.13) one finds

$$P^{(0,1)}_{q\to qg} = \frac{2}{s_{ij}} \not p \left[\frac{2z}{1-z} + (1-\rho\varepsilon)(1-z) \right],$$

$$
\begin{aligned}
P^{(0,1)}_{g\to gg} &= \frac{2}{s_{ij}}\left[-g^{\mu\nu}\left(\frac{2z}{1-z}+\frac{2(1-z)}{z}\right)-4(1-\rho\varepsilon)z(1-z)\frac{k_\perp^\mu k_\perp^\nu}{k_\perp^2}\right],\\
P^{(0,1)}_{g\to q\bar q} &= \frac{2}{s_{ij}}\left[-g^{\mu\nu}+4z(1-z)\frac{k_\perp^\mu k_\perp^\nu}{k_\perp^2}\right]. \qquad (4.16)
\end{aligned}
$$

One-loop primitive amplitudes factorize according to [27]–[31]

$$
\begin{aligned}
A_n^{(1)}(\ldots,p_i,p_j,\ldots) &= \sum_\lambda \mathrm{Split}^{(0,1)}(p_i,p_j)\; A_{n-1}^{(1)}(\ldots,p,\ldots)+\\
&\quad+\sum_\lambda \mathrm{Split}^{(1,1)}(p_i,p_j)\; A_{n-1}^{(0)}(\ldots,p,\ldots)\,. \qquad (4.17)
\end{aligned}
$$

Here, the splitting amplitudes $\mathrm{Split}^{(1,1)}$ occur as a new structure. For the one-loop amplitude $A_3^{(1)}$ for $e^+e^- \to qg\bar q$ we only need the $q\to qg$ splitting function. This function is given by [30, 31]

$$
\begin{aligned}
\mathrm{Split}^{(1,1)}_{q\to qg} &= S_\varepsilon^{-1}c_\Gamma\left(\frac{-s_{12}}{\mu^2}\right)^{-\varepsilon}\left((f_1(z)+f_2)\,\mathrm{Split}^{(0,1)}_{q\to qg}-f_2\frac{2p_i\cdot\varepsilon(p_j)}{s_{ij}^2}\bar u(p_i)\not p_j u(p)-\right)-\\
&\quad-\frac{11}{6\varepsilon}\,\mathrm{Split}^{(0,1)}_{q\to qg}\,, \qquad (4.18)
\end{aligned}
$$

where the coefficients $f_1(z)$ and f_2 are

$$
\begin{aligned}
f_1(z) &= -\frac{1}{\varepsilon^2}\,{}_2F_1\left(1,-\varepsilon,1-\varepsilon;\frac{-z}{1-z}\right),\\
f_2 &= \frac{1-\rho\varepsilon}{2(1-\varepsilon)(1-2\varepsilon)}\,. \qquad (4.19)
\end{aligned}
$$

The ε-expansion of the hypergeometric function is

$$
{}_2F_1\left(1,-\varepsilon,1-\varepsilon;\frac{-z}{1-z}\right)=1-\sum_{k=1}^{\infty}\varepsilon^k\,\mathrm{Li}_k\left(\frac{-z}{1-z}\right). \qquad (4.20)
$$

The hypergeometric function can also be written as

$$
{}_2F_1\left(1,-\varepsilon,1-\varepsilon;\frac{-z}{1-z}\right)=\Gamma(1+\varepsilon)\Gamma(1-\varepsilon)\left(\frac{z}{1-z}\right)^{\varepsilon}+1-z^{\varepsilon}\,{}_2F_1\left(\varepsilon,\varepsilon,1+\varepsilon;1-z\right), \qquad (4.21)
$$

which is useful when studying the soft limit. The interference term of $\mathrm{Split}^{(0,1)}$ with $\mathrm{Split}^{(1,1)}$ is given by

$$
\begin{aligned}
P^{(1,1)}_{q\to qg} &= \sum_\lambda u(p)\left(\mathrm{Split}^{(0,1)}_{q\to qg}\right)^*\mathrm{Split}^{(1,1)}_{q\to qg}\,\bar u(p)+c.c. \qquad (4.22)\\
&= S_\varepsilon^{-1}c_\Gamma\left(\frac{-s_{12}}{\mu^2}\right)^{-\varepsilon}\left\{f_1(z)\;P^{(0,1)}_{q\to qg}+f_2\,\frac{2}{s_{ij}}\not p\,[1-\rho\varepsilon(1-z)]\right\}-\frac{11}{6\varepsilon}P^{(0,1)}_{q\to qg}\;+c.c.
\end{aligned}
$$

Here "c.c." denotes the complex conjugate.

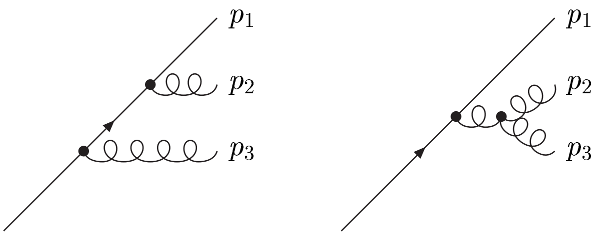

Figure 4: Diagrams contributing to the triple collinear limit $q \to qgg$.

For tree amplitudes we also have to consider the case, where three partons become collinear simultaneously. In the triple collinear limit we parametrize the momenta as follows [24]:

$$p_i = z_i p + k_{\perp i} - \frac{k_{\perp i}^2}{z_i} \frac{n}{2pn}, \tag{4.23}$$

where $n^2 = 0$, $2k_{\perp i}p = 2k_{\perp i}n = 0$, and the z_i and $k_{\perp i}$ satisfy

$$\sum z_i = 1, \qquad \sum k_{\perp i} = 0. \tag{4.24}$$

The following formulae are useful:

$$z_i = \frac{2p_i n}{2pn} = \frac{2p_i n}{2p_{ijk}n}, \qquad s_{ij} = -z_i z_j \left(\frac{k_{\perp i}}{z_i} - \frac{k_{\perp j}}{z_j} \right)^2, \qquad k_{\perp i}^2 = z_i \left(s_{jk} - (1-z_i) s_{ijk} \right). \tag{4.25}$$

Here, $p_{ijk} = p_i + p_j + p_k$. For the example of the leading N_c contributions to $e^+e^- \to qgg\bar{q}$ we have to consider only the triple collinear limit $q \to qgg$. In the triple collinear limit the tree amplitude factorizes according to

$$A_n^{(0)}(p_1, p_2, p_3, \ldots) = \sum_\lambda \text{Split}_{q\to qgg}^{(0,2)}(p_1, p_2, p_3)\, A_{n-2}^{(0)}(p, \ldots). \tag{4.26}$$

Figure 4 shows the Feynman diagrams contributing to the triple collinear limit $q \to qgg$. We are interested in the square of the splitting function $\text{Split}_{q\to qgg}^{(0,2)}$, defined by

$$P^{(0,2)} = \sum_{\lambda,\lambda'} u(p)\ \text{Split}^{(0,2)\,*}\, \text{Split}^{(0,2)}\, \bar{u}(p) \qquad \text{for quarks.} \tag{4.27}$$

For the splitting $q \to qgg$ I find

$$\begin{aligned} P_{q\to qgg}^{(0,2)} = \frac{4}{s_{123}^2} \not{p} \Bigg\{ & 2(1-\rho\varepsilon) \frac{(z_3 - (1-x_1)z_{23})^2}{x_2^2 z_{23}^2} + 4(1-\rho\varepsilon)\frac{x_1}{x_2} + (1-\rho\varepsilon)^2 \frac{x_2}{x_1} + \\ & +3 - 4\rho\varepsilon + \rho^2\varepsilon^2 + \frac{1}{x_1 x_2} \left[\frac{1+(1-z_3)^2 - \rho\varepsilon z_3^2}{z_{23}} + \frac{1+(1-z_{23})^2 - \rho\varepsilon z_{23}^2}{z_3} \right] + \\ & + \frac{1}{x_1} \Bigg[\frac{2}{z_3 z_{23}} - \frac{2}{z_3} - \frac{2}{z_{23}} - \frac{1+(1-z_3)^2 - \rho\varepsilon z_3^2}{z_{23}} + (1-\rho\varepsilon)\frac{z_3}{z_{23}} + \\ & + (1-\rho\varepsilon)\frac{z_{23}}{z_3} + \rho\varepsilon(1-\rho\varepsilon)(z_{23} - z_3) - 1 + 3\rho\varepsilon \Bigg] + \end{aligned}$$

$$+\frac{1}{x_2}\Bigg[-2\frac{1+(1-z_3)^2-\rho\varepsilon z_3^2}{z_{23}}+\frac{1+(1-z_{23})^2-\rho\varepsilon z_{23}^2}{z_3}-\frac{4}{z_{23}}-$$
$$-4\rho\varepsilon\frac{z_3}{z_{23}}-3(1-\rho\varepsilon)(z_{23}-z_3)+2+4\rho\varepsilon\Bigg]\Bigg\}. \qquad (4.28)$$

Here $x_1 = s_{12}/s_{123}$, $x_2 = s_{23}/s_{123}$ and $z_{23} = z_2 + z_3$. Note that eq. (4.28) gives the triple collinear splitting function for colour-ordered amplitudes. This implies a fixed cyclic ordering for the two gluons. In contrast, the triple collinear splitting functions given in ref. [22] and ref. [24] are the ones for the full matrix element squared, e.g. they include a sum over the permutations of the two gluons.

4.3 Factorizable double unresolved contributions

For the double unresolved contributions there are also two configurations, which have quite a simple factorizable structure in the singular limit. These two cases are: a) the emission of two independent pairs of collinear particles and b) the emission of a soft gluon together with one pair of collinear particles. Let us first consider the case of the emission of two independent pairs of collinear particles. I consider the case $p_1||p_2$ and $p_3||p_4$. The amplitude factorizes according to

$$A_n^{(0)}(p_1,p_2,p_3,p_4) = \sum_{\lambda,\lambda'} \mathrm{Split}^{(0,1)}(p_1,p_2)\,\mathrm{Split}^{(0,1)}(p_3,p_4)\,A_{n-2}^{(0)}(p,q)\,, \qquad (4.29)$$

where $p = p_1+p_2+O(k_\perp^2)$ and $q = p_3+p_4+O(k_\perp^2)$. For the leading-colour contributions to $e+e- \to qgg\bar{q}$ only the splittings $q \to qg$ and $\bar{q} \to \bar{q}g$ are relevant. Squaring the splitting functions in eq. (4.29) one obtains

$$P^{(0,2)}_{q\to qg,\bar{q}\to\bar{q}g} = P^{(0,1)}_{q\to qg}\,P^{(0,1)}_{\bar{q}\to\bar{q}g}\,. \qquad (4.30)$$

In the case of the emission of a soft gluon together with a pair of collinear partons it is for our purposes sufficient to consider the case of a soft gluon (p_3) together with a collinear splitting $q \to qg$ of the partons p_1 and p_2. I simply quote the singular factor for the squared matrix element [22, 24]:

$$P^{(0,2)}_{q\to q_1 g_2, g_3\,\mathrm{soft}} 4\frac{s_{p4}}{s_{p3}s_{34}}\,P^{(0,1)}_{q\to qg}\,. \qquad (4.31)$$

Here, $p = p_1 + p_2 + O(k_\perp^2)$ and the singular factor factorizes into an eikonal factor times a splitting function.

5. Subtraction terms

In this section I first review the subtraction terms for single unresolved contributions. I then give the subtraction terms for double unresolved contributions and for contributions with one virtual and one real unresolved parton.

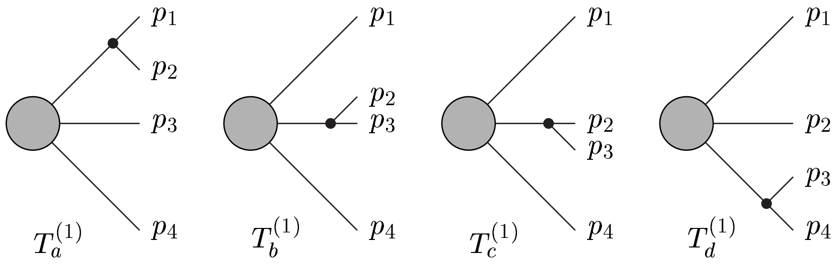

Figure 5: Splitting topologies for NLO subtraction. At each splitting the emitted particle is directed towards the spectator.

5.1 Subtraction terms for single unresolved contributions

The approxiamtion term $d\alpha_{n+1}^{(0,1)}$ is given as a sum over dipoles [15]:

$$d\alpha_{n+1}^{(0,1)} = \sum_{\text{topologies } T} \mathcal{D}_{n+1}^{(0,1)}(T)\,. \tag{5.1}$$

For leading colour contributions the sum reduces to combinations where the emitting parton i and the emitted parton j are adjacent, and the spectator k is either adjacent to i or j. It is also convenient to denote by k' the other parton adjacent to the pair i and j. Therefore the cyclic ordering is either k', i, j, k or k, j, i, k'. The relevant topologies for the example of the leading N_c contributions to $e^+e^- \to qgg\bar{q}$ are shown in figure 5. A dipol is constructed from amputated amplitudes, evaluated with mapped momenta and dipole splitting functions. By an amputated amplitude I mean an amplitude where a polarization vector (or external spinor for fermions) has been removed. I denote amputated amplitudes by $|A\rangle$. If the amputated parton is a gluon, one has

$$\left| A_n^{(0)}(\dots,p,\dots) \right\rangle = \frac{\partial}{\partial \varepsilon_\mu(p)} A_n^{(0)}(\dots,p,\dots)\,. \tag{5.2}$$

If the amputated parton is a quark, one has

$$\left| A_n^{(0)}(\dots,p,\dots) \right\rangle = \frac{\partial}{\partial \bar{u}(p)} A_n^{(0)}(\dots,p,\dots)\,. \tag{5.3}$$

The dipole subtraction terms are obtained by sandwiching a dipole splitting function in between these amputated amplitudes:

$$\mathcal{D}_{n+1}^{(0,1)}(T) = c \left\langle A_{n+1}^{(0)}(\dots,p_e,\dots) \right| \mathcal{P}^{(0,1)}(T) \left| A_{n+1}^{(0)}(\dots,p_e,\dots) \right\rangle d\phi_{n+2}\,. \tag{5.4}$$

Here, c denotes an overall prefactor. For the example $e^+e^- \to qgg\bar{q}$, it is given by

$$c = (4\pi\alpha)^2 (4\pi\alpha_s)^2 |c_0|^2 \times \frac{1}{4}\left(N_c^2 - 1\right) N_c\,. \tag{5.5}$$

The dipole splitting functions read

$$\mathcal{P}_{q\to qg}^{(0,1)}(p_k', p_i, p_j, p_k) = \frac{2}{s_{ijk}} \frac{1}{y} \not{p}_e \left[\frac{2}{1 - z(1-y)} - 2 + (1 - \rho\varepsilon)(1-y)(1-z) \right],$$

$$
\begin{aligned}
\mathcal{P}^{(0,1)}_{g\to gg}(p_k', p_i, p_j, p_k) &= \frac{2}{s_{ijk}}\frac{1}{y}\left[-g^{\mu\nu}\left(\frac{2}{1-z(1-y)}-2\right)+(1-\rho\varepsilon)\,\frac{4r(1-y)^2}{s_{ij}}k_\perp^\mu k_\perp^\nu\right],\\
\mathcal{P}^{(0,1)}_{g\to q\bar{q}}(p_k', p_i, p_j, p_k) &= \frac{2}{s_{ijk}}\frac{1}{y}\left[-\frac{1}{2}g^{\mu\nu}-\frac{4r(1-y)^2}{s_{ij}}k_\perp^\mu k_\perp^\nu\right],
\end{aligned}
\tag{5.6}
$$

where

$$
r = \frac{2p_{k'}p_e}{2p_{k'}p_e + 2p_e p_s}\,. \tag{5.7}
$$

These splitting functions differ by finite terms from the original choice of Catani and Seymour. The new set of splitting functions in eq. (5.6) is introduced for later convenience with the NNLO subtraction terms. Although arbitrary non-singular terms may be added to the NLO subtraction terms, the NNLO subtraction terms depend on the actual form of the NLO subtraction terms.

The dipole splitting functions in eq. (5.6) have also the correct singular behaviour in the soft limit. The term $1/(1-z(1-y))$ equals $s_{ijk}/(s_{ij}+s_{jk})$ and combining this term with the one from the dipole, where the rôles of emitter and spectator are exchanged, one obtains

$$
\frac{1}{s_{ij}}\frac{s_{ijk}}{(s_{ij}+s_{jk})}+\frac{1}{s_{jk}}\frac{s_{ijk}}{(s_{ij}+s_{jk})}=\frac{s_{ijk}}{s_{ij}s_{jk}}\,, \tag{5.8}
$$

which is the eikonal factor in the soft limit.

For the $q \to qg$ splitting, the quark polarization matrix $\not{p}_e$ is included in the definition of the dipole splitting function. For the $g \to gg$ and $g \to q\bar{q}$ splittings, the full splitting function is shared between the dipoles with spectators k and k', where k and k' are the partons adjacent to the (i,j) pair in the colour ordered amplitude $A^{(0)}(\ldots,k',i,j,k,\ldots)$. For the $g \to gg$ splitting the dipole splitting function is chosen in such a way that the singularity when parton j becomes soft resides in $\mathcal{P}_{g\to gg}(p_i,p_j;p_k)$, where as the singularity when parton i becomes soft resides in $\mathcal{P}_{g\to gg}(p_j,p_i;p_{k'})$. The amputated amplitudes are evaluated with mapped momenta. The mapping of momenta relates a $n+1$ parton configuration to a n parton configuration:

$$
P_{n+1\to n} : (p_1,\ldots,p_{n+1}) \to \left(p_1',\ldots,p_n'\right). \tag{5.9}
$$

Such a mapping has to satisfy momentum conservation and the on-mass-shell conditions. Furthermore it must have the right behaviour in the singular limits. Several choices for such a mapping exist [15, 37]. One possible choice relates three parton momenta of the $n+1$ parton configuration to two parton momenta of the n parton configuration, while leaving the remaining $n-2$ parton momenta unchanged. This mapping is given by

$$
\begin{aligned}
p_e &= p_i + p_j - \frac{y}{1-y}p_k\,,\\
p_s &= \frac{1}{1-y}p_k\,.
\end{aligned}
\tag{5.10}
$$

p_i, p_j and p_k are the momenta of the $n+1$ parton configuration, and p_e and p_s are the resulting momenta of the of the n parton configuration. The variables y and z are given by

$$y = \frac{s_{ij}}{s_{ijk}}, \qquad z = \frac{s_{ik}}{s_{ik}+s_{jk}}. \tag{5.11}$$

It is easily verified that this mapping satisfies momentum conservation

$$p_e + p_s = p_i + p_j + p_k, \tag{5.12}$$

and the on-shell conditions

$$p_e^2 = 0, \qquad p_s^2 = 0. \tag{5.13}$$

The singular limit corresponds to $y \to 0$ and in this limit we have

$$p_e - (p_i + p_j) = O(y) \qquad , \qquad p_s - p_k = O(y). \tag{5.14}$$

It is also useful to introduce the vector $k_\perp$, defined by

$$k_\perp = (1-z)p_i - zp_j - (1-2z)\,\frac{y}{1-y} p_k. \tag{5.15}$$

With this vector and the value of the longitudinal momentum fraction z, the mapping may be inverted to yield

$$\begin{aligned} p_i &= zp_e + k_\perp + y\,(1-z)\,p_s, \\ p_j &= (1-z)p_e - k_\perp + yzp_s, \\ p_k &= (1-y)p_s, \end{aligned} \tag{5.16}$$

where y is now given by

$$y = \frac{-k_\perp^2}{z(1-z)\,(p_e+p_s)^2}. \tag{5.17}$$

The singular limit occurs now for $k_\perp \to 0$. The $g \to gg$ and $g \to q\bar{q}$ splittings also involve spin correlation through the spin correlation tensor $k_\perp^\mu k_\perp^\nu$. Note that the spin correlation tensor $k_\perp^\mu k_\perp^\nu$ is orthogonal to p_e and p_s:

$$2p_e k_\perp = 2p_s k_\perp = 0. \tag{5.18}$$

Further note that the contraction of p_e into an amputated amplitude vanishes due to gauge invariance.

The parameter ρ specifies the variant of dimensional regularization: $\rho = 1$ corresponds to the CDR/HV schemes and $\rho = 0$ to a four-dimensional scheme.

5.2 Subtraction terms for double unresolved contributions

In this subsection 1 give the subtraction terms for double unresolved contributions. To be more precise, these subtraction terms have to approximate

$$\mathcal{O}_{n+2}\, d\sigma_{n+2}^{(0)} - \mathcal{O}_{n+1} \circ d\alpha_{n+1}^{(0,1)} \tag{5.19}$$

in all single and double unresolved limits. Eq. (5.19) is integrable over single unresolved regions for observables, which vanish on n-parton configurations, e.g. $(n+1)$-jet observables. However, as discussed in the example of eq. (2.15), for a generic n-jet observable, eq. (5.19) is in general not integrable over single unresolved regions. Therefore, not only the singularities corresponding to double unresolved limits, but also the singularities corresponding to single unresolved limits have to be subtracted.

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The subtraction term $d\alpha_n^{(0,2)}$ at NNLO is also written as a sum over topologies:

$$d\alpha_n^{(0,2)} = \sum_{\text{topologies } T} \mathcal{D}_n^{(0,2)}(T)\,. \tag{5.20}$$

These topologies should be thought of as pictorial representations of the pole structure of the subtraction terms $\mathcal{D}_n^{(0,2)}(T)$ and the way the momenta of the n-parton configurations are constructed from the momenta of the $(n+2)$-parton configurations. The intuitive picture of "iterated dipoles" is only justified for the approximations of $d\alpha_{n+1}^{(0,1)}$. The relevant topologies for the leading N_c contributions to $e^+e^- \to qgg\bar{q}$ are shown in figure 6 to figure 7. They fall into two classes: two single splittings from two different branches (topologies $T_a^{(2)}$ and $T_b^{(2)}$) and two sequential splittings from the same branch (topologies $T_c^{(2)}$- $T_h^{(2)}$). It is sufficient to consider topologies $T_a^{(2)}$ and $T_c^{(2)}$-$T_e^{(2)}$ only. The remaining topologies are related by symmetry to the ones mentioned above. The subtraction term is given by

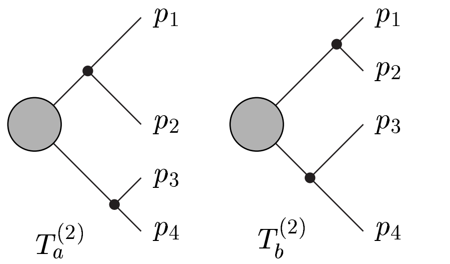

Figure 6: Splitting topologies for NNLO subtraction: Two single splittings from two different branches. At each splitting the emitted particle is directed towards the spectator.

$$\begin{aligned}\mathcal{D}_n^{(0,2)}(T) &= c\mathcal{J}(T) \times \\ &\times \left\langle A_n^{(0)}\left(\ldots,p_e^{(2)},\ldots,p_s^{(2)},\ldots\right)\right| \mathcal{P}^{(0,2)}(T) \left| A_n^{(0)}\left(\ldots,p_e^{(2)},\ldots,p_s^{(2)},\ldots\right)\right\rangle d\phi_{n+2}\,.\end{aligned} \tag{5.21}$$

$\mathcal{J}$ is a prefactor not singular in any limit, which can be used to absorbe jacobians into the definition of the subtraction terms, thus rendering the analytical integration over the unresolved phase space simpler. Here, I simply take $\mathcal{J} = 1$. The splitting function $\mathcal{P}^{(0,2)}(T)$ is sandwiched between double-amputated amplitudes for the topologies $T_a^{(2)}$ and $T_b^{(2)}$, and

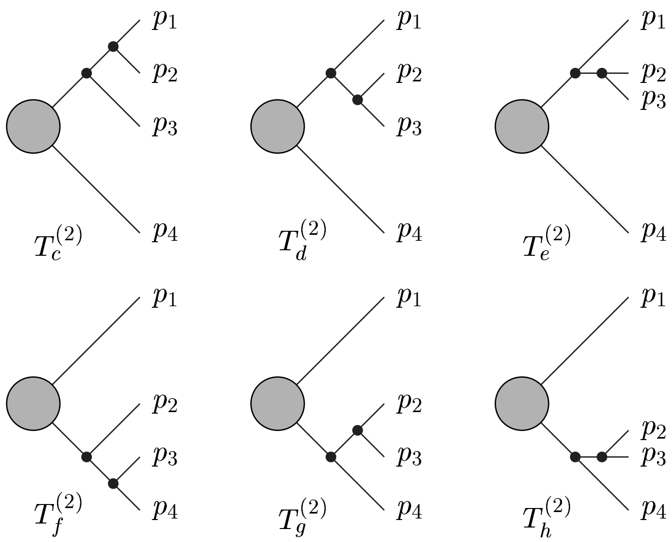

Figure 7: Splitting topologies for NNLO subtraction: Two sequential splittings from the same branch. At each splitting the emitted particle is directed towards the spectator.

is sandwiched between single-amputated amplitudes for the topologies $T_c^{(2)}$-$T_h^{(2)}$. It is convenient to write

$$\mathcal{P}^{(0,2)}(T) = \mathcal{P}^{(0,2)}_{(0,0)}(T) - \mathcal{P}^{(0,2)}_{(0,1)}(T), \tag{5.22}$$

where $\mathcal{P}^{(0,2)}_{(0,0)}$ is an approxiamation to $d\sigma^{(0)}_{n+2}$ and $\mathcal{P}^{(0,2)}_{(0,1)}$ is an approxiamation to $d\alpha^{(0,1)}_{n+1}$. The notation with super- and subscripts is as follows:

$$\mathcal{P}^{(l,k)}_{(l',k')}(T) \tag{5.23}$$

contributes to $d\alpha^{(l,k)}$ and approximates a term of $d\alpha^{(l',k')}$, (with $d\sigma^{(l)}$ identified as $d\alpha^{(l,0)}$). It should be noted that the subtraction terms are not unique. For example, the complete triple collinear splitting function $P_{q\to qgg}$ is shared between the topologies $T_c^{(2)}$, $T_d^{(2)}$ and $T_e^{(2)}$. Therefore, a term of the form

$$\frac{c}{s_{123}^2}, \tag{5.24}$$

which is only singular in the triple collinear limit and finite in all other cases, can be freely moved between these topologies. On the other hand, a term of the form

$$\frac{c'}{s_{12}s_{123}}, \tag{5.25}$$

which is singular in the triple collinear limit and in the single unresolved limit $s_{12} \to 0$ is uniquely associated with topology $T_c^{(2)}$, as the pictorial representation suggests. Below I will give one possible set for the subtraction terms.

I briefly mention how the n-parton momentum configuration is constructed from the $(n+2)$-parton configuration. As an example I discuss topology $T_c^{(2)}$. For the outermost splitting, particle 1 is the emitter, particle 2 is the emitted soft or collinear particle and particle 3 is the spectator. Particle 4 is not involved in this splitting. From this 4-parton momentum configuration (p_1, p_2, p_3, p_4) a 3-parton momentum configuration $(p_e^{(1)}, p_s^{(1)}, p_4)$ is first constructed, using the Catani-Seymour reconstruction function given in eq. (5.10). From the 3-parton momentum configuration a 2-parton momentum configuration $(p_e^{(2)}, p_s^{(2)})$ is then constructed, using the symmetric reconstruction functions given by Kosower [37]. This symmetric choice is necessary to ensure that topologies $T_c^{(2)}$ and $T_b^{(2)}$ are evaluated with the same n-parton momentum configuration. This is necessary for a cancellation of (subleading) single unresolved singularities in $s_{12} \to 0$ and is discussed in detail in section 6.

To present the splitting functions for double unresolved configurations I use the following notation:

$$\bar{y} = 1 - y\,, \qquad \bar{z} = 1 - z\,, \tag{5.26}$$

and $p_{ij} = p_i + p_j$.

5.2.1 Topology A

This topology corresponds to two independent single splittings and the splitting function $\mathcal{P}$ is inserted into a double-amputated amplitude. The splitting functions read:

$$\mathcal{P}^{(0,2)}_{(0,0)\,q\to qg,\bar{q}\to\bar{q}g}\left(T_a^{(2)}\right) = \left(\not{p}_e^{(2)}\right)_{ij}\left(\not{p}_s^{(2)}\right)_{\bar{i}\bar{j}}\frac{4}{s_{1234}^2}\frac{1}{y_1 x_2\left(\bar{y}_1 x_1 + y_1 x_2\right)} \times$$

$$\times\left\{\left[1+\frac{\bar{y}_1\bar{z}_1}{u}\right]\frac{2}{\bar{y}_1x_1+(y_1+\bar{y}_1\bar{z}_1)\,x_2}+\left[\frac{1}{y_1x_2+u}+\frac{1}{y_1+\bar{y}_1\bar{z}_1}\right]\times\right.$$
$$\times\left[-2+(1-\rho\varepsilon)x_2\right]+\left[\frac{1}{\bar{y}_1x_1+x_2}+\frac{u}{\bar{y}_1x_1+\bar{y}_1\bar{z}_1x_2}\right]\times$$
$$\times\left[-2+(1-\rho\varepsilon)u\right]+\left[-2+(1-\rho\varepsilon)x_2\right]\left[-2+(1-\rho\varepsilon)u\right]-$$
$$\left.-\rho\varepsilon\left(\bar{y}_1x_1+y_1x_2\right)\left[\frac{2}{y_1+\bar{y}_1\bar{z}_1}-2+(1-\rho\varepsilon)\bar{y}_1\bar{z}_1\right]\right\},$$
$$\mathcal{P}^{(0,2)}_{(0,1)\,q\to qg,\bar{q}\to\bar{q}g}\left(T_a^{(2)}\right)=\left(\not{p}_e^{(2)}\right)_{ij}\left(\not{p}_s^{(2)}\right)_{\bar{i}\bar{j}}\frac{4}{s_{1234}^2}\frac{1}{x_1x_2}\left[\frac{2}{x_1+x_2}-2+(1-\rho\varepsilon)\,x_2-\rho\varepsilon x_1\right]\times$$
$$\times\frac{1}{y_1}\left[\frac{2}{1-z_1(1-y_1)}-2+(1-\rho\varepsilon)\,(1-y_1)(1-z_1)\right]. \tag{5.27}$$

As already noted, this is inserted into a double-amputated amplitude. The variables x_1, x_2 and x_3 are given by

$$x_1=y_2\,,\qquad x_2=(1-y_2)(1-z_2)\,,\qquad x_3=(1-y_2)z_2\,. \tag{5.28}$$

The variables y_1, z_1, y_2 and z_2 are given by

$$y_1=\frac{s_{34}}{s_{234}}\,,\qquad z_1=\frac{s_{24}}{s_{23}+s_{24}}\,,$$
$$y_2=\frac{1}{(1-y_1)}\frac{s_{12}}{s_{1234}}\,,\qquad z_2=1-\frac{1}{(1-y_2)}\frac{s_{234}}{s_{1234}}\,. \tag{5.29}$$

The remaining variable u is given by

$$u=(z_1+\bar{y}_1\bar{z}_1)\,x_1+\bar{y}_1\bar{z}_1x_2+\bar{z}_1x_3+2(1-2w)\sqrt{y_1z_1\bar{z}_1x_1x_3}\,,$$
$$w=\frac{1}{2}\,(1-c_2)\,,$$
$$c_2=\frac{s_{12}s_{23}^2+s_{24}\left((s_{23}+s_{24})(s_{13}+s_{14})-s_{12}s_{23}\right)-s_{14}(s_{23}+s_{24})^2}{2s_{23}\sqrt{s_{12}s_{24}\left((s_{23}+s_{24})(s_{13}+s_{14})-s_{12}s_{23}\right)}}\,. \tag{5.30}$$

The amputated amplitude is evaluated with the mapped momenta

$$p_e^{(2)}=\frac{1}{2}\left[1+q+\frac{\bar{y}_2\bar{z}_2\,(1+q-2r)}{1-\bar{y}_2\bar{z}_2}\right]p_1+rp_s^{(1)}+\frac{1}{2}\left[1-q+\frac{y_2\,(1-q-2r)}{1-y_2}\right]p_e^{(1)},$$
$$p_s^{(2)}=\frac{1}{2}\left[1-q-\frac{\bar{y}_2\bar{z}_2\,(1+q-2r)}{1-\bar{y}_2\bar{z}_2}\right]p_1+(1-r)p_s^{(1)}+\frac{1}{2}\left[1+q-\frac{y_2\,(1-q-2r)}{1-y_2}\right]p_e^{(1)}\,, \tag{5.31}$$

where

$$p_e^{(1)}=p_{34}-\frac{y_1}{1-y_1}p_2\,,\qquad p_s^{(1)}=\frac{1}{1-y_1}p_2\,, \tag{5.32}$$

and the variables r and q are given by

$$r=\frac{\bar{y}_2\bar{z}_2}{y_2+\bar{y}_2\bar{z}_2}\,,\qquad q=\sqrt{1+4r(1-r)y_2\frac{1-z_2}{z_2}}\,. \tag{5.33}$$

5.2.2 Topology C

The splitting functions read:

$$
\begin{aligned}
\mathcal{P}^{(0,2)}_{(0,0)\,q\to qgg}\left(T^{(2)}_{c}\right) = \not{p}^{(2)}_{e}\frac{4}{s^2_{1234}}\frac{1}{x_1^2}\times \\
\times\Bigg\{\frac{1}{y_1\left(y_1+\bar{y}_1\bar{z}_1\right)}\Bigg[\frac{2}{\left(y_1+\bar{y}_1\bar{z}_1\right)x_1+\bar{y}_1x_2}+\frac{2}{y_1x_1+u}+ \\
+\frac{u}{\left(y_1+\bar{y}_1\bar{z}_1\right)x_1+\bar{y}_1x_2}\left(-2+(1-\rho\varepsilon)u\right)+ \\
+\frac{\bar{y}_1x_2}{u}\left(-2+(1-\rho\varepsilon)\bar{y}_1x_2\right)\Bigg] \\
+\frac{1}{y_1}\Bigg[\frac{2}{u\left[\left(y_1+\bar{y}_1\bar{z}_1\right)x_1+\bar{y}_1x_2\right]}+\frac{1}{x_1+\bar{y}_1x_2}\left(-2+(1-\rho\varepsilon)u\right)- \\
-\frac{4}{u}+(1-\rho\varepsilon)\frac{\bar{y}_1x_2}{u}-\frac{\bar{y}_1x_2}{u}\left(-2+(1-\rho\varepsilon)\bar{y}_1x_2\right)+ \\
+\rho\varepsilon(1-\rho\varepsilon)u-\rho\varepsilon(1-\rho\varepsilon)\bar{y}_1x_2-1+3\rho\varepsilon\Bigg]+(1-\rho\varepsilon)^2\,\frac{\bar{y}_1\bar{z}_1}{y_1}+ \\
+3-4\rho\varepsilon+\rho^2\varepsilon^2+\frac{x_1}{y_1}\Bigg[-\rho\varepsilon\left(\frac{2}{y_1+\bar{y}_1\bar{z}_1}-2+(1-\rho\varepsilon)\bar{y}_1\bar{z}_1\right)+\frac{2}{u}- \\
-(1-\rho\varepsilon)\left(\frac{x_1}{u}-2\bar{y}_1z_1+(1-\rho\varepsilon)\bar{y}_1z_1x_1\right)+ \\
+\frac{1}{u\left(y_1+\bar{y}_1\bar{z}_1\right)}\left(-2+(1-\rho\varepsilon)x_1\right)\Bigg]\Bigg\}, \\
\mathcal{P}^{(0,2)}_{(0,1)\,q\to qgg}\left(T^{(2)}_{c}\right) = \not{p}^{(2)}_{e}\frac{4}{s^2_{1234}}\frac{1}{x_1^2}\left[\frac{2}{x_1+x_2}-2+(1-\rho\varepsilon)\,x_2-\rho\varepsilon x_1\right]\times \\
\times\frac{1}{y_1}\left[\frac{2}{1-z_1(1-y_1)}-2+(1-\rho\varepsilon)\,(1-y_1)(1-z_1)\right].
\end{aligned}
\tag{5.34}
$$

The variables x_1, x_2 and x_3 are given by

$$
x_1 = y_2\,, \qquad x_2 = (1-y_2)(1-z_2)\,, \qquad x_3 = (1-y_2)z_2\,. \tag{5.35}
$$

The variables y_1, z_1, y_2 and z_2 are given by

$$
\begin{aligned}
y_1 &= \frac{s_{12}}{s_{123}}\,, \qquad z_1 = \frac{s_{13}}{s_{13}+s_{23}}\,, \\
y_2 &= \frac{s_{123}}{s_{1234}}\,, \qquad z_2 = 1-\frac{1}{(1-y_1)(1-y_2)}\frac{s_{34}}{s_{1234}}\,.
\end{aligned}
\tag{5.36}
$$

The remaining variable u is given by

$$
\begin{aligned}
u &= \bar{y}_1\bar{z}_1x_1+\left(z_1+\bar{y}_1\bar{z}_1\right)x_2+\bar{z}_1x_3+2(1-2w)\sqrt{y_1z_1\bar{z}_1x_2x_3}\,, \\
w &= \frac{1}{2}\left(1-c_2\right), \\
c_2 &= \frac{s_{12}s_{23}s_{34}+s_{13}\left((s_{13}+s_{23})(s_{14}+s_{24})-s_{12}s_{34}\right)-s_{14}(s_{13}+s_{23})^2}{2\sqrt{s_{12}s_{23}s_{34}s_{13}\left((s_{13}+s_{23})(s_{14}+s_{24})-s_{12}s_{34}\right)}}\,.
\end{aligned}
\tag{5.37}
$$

The amputated amplitude is evaluated with the mapped momenta

$$
\begin{aligned}
p_e^{(2)} &= \frac{1}{2}\left[1+q-\frac{\bar{y}_2\bar{z}_2\,(1-q-2r)}{1-\bar{y}_2\bar{z}_2}\right]p_e^{(1)}+(1-r)p_s^{(1)}+\frac{1}{2}\left[1-q-\frac{y_2\,(1+q-2r)}{1-y_2}\right]p_4\,,\\
p_s^{(2)} &= \frac{1}{2}\left[1-q+\frac{\bar{y}_2\bar{z}_2\,(1-q-2r)}{1-\bar{y}_2\bar{z}_2}\right]p_e^{(1)}+rp_s^{(1)}+\frac{1}{2}\left[1+q+\frac{y_2\,(1+q-2r)}{1-y_2}\right]p_4\,,
\end{aligned}
\tag{5.38}
$$

where

$$
p_e^{(1)} = p_{12}-\frac{y_1}{1-y_1}p_3\,,\qquad p_s^{(1)}=\frac{1}{1-y_1}p_3\,, \tag{5.39}
$$

and the variables r and q are given by

$$
r=\frac{y_2}{y_2+\bar{y}_2\bar{z}_2}\,,\qquad q=\sqrt{1+4r(1-r)y_2\frac{1-z_2}{z_2}}\,. \tag{5.40}
$$

5.2.3 Topology D

The splitting functions read:

$$
\begin{aligned}
\mathcal{P}^{(0,2)}_{(0,0)\,q\to qgg}\left(T_d^{(2)}\right) = \not{p}_e^{(2)}\frac{4}{s_{1234}^2}\frac{1}{x_1^2}\times&\\
\times\Bigg\{\frac{1}{y_1(y_1+\bar{y}_1\bar{z}_1)}\Bigg[&\frac{2}{(y_1+\bar{y}_1\bar{z}_1)x_1+u}+\frac{2}{(y_1+\bar{y}_1\bar{z}_1)\,x_1+x_2+y_1x_3}+\\
&+\frac{y_1x_1+x_2+y_1x_3}{(y_1+\bar{y}_1\bar{z}_1)\,x_1+u}\times\\
&\times\left[-2+(1-\rho\varepsilon)\,(y_1x_1+x_2+y_1x_3)\right]\times\\
&+\frac{u}{y_1x_1+x_2+y_1x_3}\left[-2+(1-\rho\varepsilon)u\right]\Bigg]+\\
+\frac{1}{y_1}\Bigg[&-\frac{4}{(1+y_1)x_1+x_2+y_1x_3}+\frac{2}{x_1+u}-\frac{2u}{y_1x_1+x_2+y_1x_3}\times\\
&\times\left[-2+(1-\rho\varepsilon)u\right]-4\rho\varepsilon\frac{u}{y_1x_1+x_2+y_1x_3}-\\
&-(1-\rho\varepsilon)\,(y_1x_1+x_2+y_1x_3)+3(1-\rho\varepsilon)u-2+4\rho\varepsilon\Bigg]+\\
&+4(1-\rho\varepsilon)\frac{\bar{y}_1\bar{z}_1}{y_1}+\frac{x_1}{y_1}(-\rho\varepsilon)\left[\frac{2}{y_1+\bar{y}_1\bar{z}_1}-2+\bar{y}_1^2z_1\bar{z}_1\right]\\
&+\frac{2(1-\rho\varepsilon)}{y_1^2\left[(1+y_1)x_1+x_2+y_1x_3\right]}\times\\
&\Bigg[(1-\bar{y}_1\bar{z}_1)^2\,(y_1x_1+x_2+y_1x_3)-2u\,(1-\bar{y}_1\bar{z}_1)+\\
&+\frac{u^2}{y_1x_1+x_2+y_1x_3}\Bigg]\Bigg\}\,,\\
\mathcal{P}^{(0,2)}_{(0,1)\,q\to qgg}\left(T_d^{(2)}\right) = \not{p}_e^{(2)}\frac{4}{s_{1234}^2}\Bigg\{&\frac{1}{x_1^2}\left[\frac{2}{x_1+x_2}-2+(1-\rho\varepsilon)\,x_2-\rho\varepsilon x_1\right]\frac{1}{y_1}\times\\
&\times\left[\frac{2}{1-z_1(1-y_1)}-2\right]+\frac{8\,(1-\rho\varepsilon)\,x_3}{x_1^2\,(x_1+x_2)}(1-2w)^2\times
\end{aligned}
$$

$$\times \frac{1}{y_1}(1-y_1)^2 z_1(1-z_1) +$$
$$+ \frac{2}{x_1+x_2}\left[(1-\rho\varepsilon)\left(\frac{x_2^2}{x_1^2}+1\right) - 2\rho\varepsilon\frac{x_2}{x_1}\right]\times$$
$$\times \frac{1}{y_1}(1-y_1)^2 z_1(1-z_1)\Bigg\} . \tag{5.41}$$

The variables x_1, x_2 and x_3 are given by

$$x_1 = y_2\,, \qquad x_2 = (1-y_2)(1-z_2)\,, \qquad x_3 = (1-y_2)z_2\,. \tag{5.42}$$

The variables y_1, z_1, y_2 and z_2 are given by

$$\begin{aligned} y_1 &= \frac{s_{23}}{s_{123}}\,, \qquad z_1 = \frac{s_{13}}{s_{12}+s_{13}}\,, \\ y_2 &= \frac{s_{123}}{s_{1234}}\,, \qquad z_2 = \frac{1}{(1-y_1)(1-y_2)}\frac{s_{14}}{s_{1234}}\,. \end{aligned} \tag{5.43}$$

The remaining variable u is given by

$$\begin{aligned} u &= z_1 x_2 + y_1 \bar{z}_1 x_3 - 2(1-2w)\sqrt{y_1 z_1 \bar{z}_1 x_2 x_3}\,, \\ w &= \frac{1}{2}(1-c_2)\,, \\ c_2 &= \frac{s_{12}s_{23}s_{14} + s_{13}((s_{12}+s_{13})(s_{24}+s_{34}) - s_{23}s_{14}) - s_{34}(s_{12}+s_{13})^2}{2\sqrt{s_{12}s_{23}s_{13}s_{14}((s_{12}+s_{13})(s_{24}+s_{34}) - s_{23}s_{14})}}\,. \end{aligned} \tag{5.44}$$

The amputated amplitude is evaluated with the mapped momenta

$$\begin{aligned} p_e^{(2)} &= \frac{1}{2}\left[1+q-\frac{\bar{y}_2\bar{z}_2\,(1-q-2r)}{1-\bar{y}_2\bar{z}_2}\right]p_s^{(1)} + (1-r)p_e^{(1)} + \frac{1}{2}\left[1-q-\frac{y_2\,(1+q-2r)}{1-y_2}\right]p_4\,, \\ p_s^{(2)} &= \frac{1}{2}\left[1-q+\frac{\bar{y}_2\bar{z}_2(1-q-2r)}{1-\bar{y}_2\bar{z}_2}\right]p_s^{(1)} + rp_e^{(1)} + \frac{1}{2}\left[1+q+\frac{y_2(1+q-2r)}{1-y_2}\right]p_4\,, \end{aligned} \tag{5.45}$$

where

$$p_e^{(1)} = p_{23} - \frac{y_1}{1-y_1}p_1, \qquad p_s^{(1)} = \frac{1}{1-y_1}p_1, \tag{5.46}$$

and the variables r and q are given by

$$r = \frac{y_2}{y_2+\bar{y}_2\bar{z}_2}, \qquad q = \sqrt{1+4r(1-r)y_2\frac{1-z_2}{z_2}}. \tag{5.47}$$

5.2.4 Topology E

The splitting functions read:

$$\mathcal{P}^{(0,2)}_{(0,0)\,q\to qgg}\left(T_e^{(2)}\right) = \not{p}_e^{(2)}\frac{4}{s_{1234}^2}\times$$
$$\times\Bigg\{\frac{1}{(x_1+y_1x_2+y_1x_3)y_1x_2}$$
$$\times\left[-\frac{4}{x_1+(1+y_1)\,x_2+y_1x_3}+\right.$$

$$
\begin{aligned}
&+\frac{1}{(y_1+\bar{y}_1\bar{z}_1)}\left[-2+(1-\rho\varepsilon)x_2\right]+4-2(1-\rho\varepsilon)x_2\Big]+\\
&+\frac{1}{y_1x_2[(y_1+\bar{y}_1\bar{z}_1)x_2+u]}\frac{1}{(y_1+\bar{y}_1\bar{z}_1)}\left[-2+(1-\rho\varepsilon)x_2\right]+\\
&+\frac{1}{y_1x_2}(-\rho\varepsilon)\left[\frac{2}{y_1+\bar{y}_1\bar{z}_1}-2+\bar{y}_1^2z_1\bar{z}_1\right]\Big]\Big\}\,,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{P}^{(0,2)}_{(0,1)\,q\to qgg}\left(T_e^{(2)}\right) &= \not{p}_e^{(2)}\frac{4}{s_{1234}^2}\frac{1}{x_1x_2}\left[\frac{2}{x_1+x_2}-2+(1-\rho\varepsilon)\,x_2-\rho\varepsilon x_1\right]\times\\
&\quad\times\frac{1}{y_1}\left[\frac{2}{1-z_1(1-y_1)}-2\right].
\end{aligned}
\tag{5.48}
$$

The variables x_1, x_2 and x_3 are given by

$$
x_1=y_2\,,\qquad x_2=(1-y_2)(1-z_2)\,,\qquad x_3=(1-y_2)z_2\,. \tag{5.49}
$$

The variables y_1, z_1, y_2 and z_2 are given by

$$
\begin{aligned}
y_1 &= \frac{s_{23}}{s_{234}}\,,\qquad z_1=\frac{s_{24}}{s_{24}+s_{34}},\\
y_2 &= \frac{1}{(1-y_1)}\frac{(s_{123}s_{234}-s_{23}s_{1234})}{s_{234}s_{1234}}\,,\qquad z_2=1-\frac{1}{(1-y_2)}\frac{s_{234}}{s_{1234}}\,.
\end{aligned}
\tag{5.50}
$$

The remaining variable u is given by

$$
\begin{aligned}
u &= z_1x_1+y_1\bar{z}_1x_3-2(1-2w)\sqrt{y_1z_1\bar{z}_1x_1x_3}\,,\\
w &= \frac{1}{2}\left(1-c_2\right),\\
c_2 &= \frac{s_{23}s_{34}s_{14}+s_{24}((s_{12}+s_{13})(s_{24}+s_{34})-s_{23}s_{14})-s_{12}(s_{24}+s_{34})^2}{2\sqrt{s_{23}s_{34}s_{14}s_{24}((s_{12}+s_{13})(s_{24}+s_{34})-s_{23}s_{14})}}\,.
\end{aligned}
\tag{5.51}
$$

The amputated amplitude is evaluated with the mapped momenta

$$
\begin{aligned}
p_e^{(2)} &= \frac{1}{2}\left[1+q+\frac{\bar{y}_2\bar{z}_2(1+q-2r)}{1-\bar{y}_2\bar{z}_2}\right]p_1+rp_e^{(1)}+\frac{1}{2}\left[1-q+\frac{y_2(1-q-2r)}{1-y_2}\right]p_s^{(1)}\,,\\
p_s^{(2)} &= \frac{1}{2}\left[1-q-\frac{\bar{y}_2\bar{z}_2(1+q-2r)}{1-\bar{y}_2\bar{z}_2}\right]p_1+(1-r)p_e^{(1)}+\frac{1}{2}\left[1+q-\frac{y_2(1-q-2r)}{1-y_2}\right]p_s^{(1)}\,,
\end{aligned}
\tag{5.52}
$$

where

$$
p_e^{(1)}=p_{23}-\frac{y_1}{1-y_1}p_4\,,\qquad p_s^{(1)}=\frac{1}{1-y_1}p_4\,, \tag{5.53}
$$

and the variables r and q are given by

$$
r=\frac{\bar{y}_2\bar{z}_2}{y_2+\bar{y}_2\bar{z}_2}\,,\qquad q=\sqrt{1+4r(1-r)y_2\frac{1-z_2}{z_2}}\,. \tag{5.54}
$$

5.3 Subtraction terms for contributions with one virtual and one real unresolved parton

In this section 1 give the subtracttion term $d\alpha_n^{(1,1)}$, which approximates

$$d\sigma_{n+1}^{(1)} + d\alpha_{n+1}^{(0,1)} = (4\pi\alpha)^2 (4\pi\alpha_s) \frac{\alpha_s}{2\pi} |c_0|^2 \frac{1}{4} \left(N_c^2 - 1\right) N_c \times \\ \times \left[A_{n+1}^{(0)}{}^* A_{n+1}^{(1)} + A_{n+1}^{(1)}{}^* A_{n+1}^{(0)} + A_{n+1}^{(0)}{}^* \mathcal{I} A_{n+1}^{(0)} \right] d\phi_{n+1} \qquad (5.55)$$

$\mathcal{I}$ is obtained from integrating $d\alpha_{n+1}^{(0,1)}$ over the unresolved phase space and is given for our example by

$$\mathcal{I} = S_\varepsilon^{-1} \frac{(4\pi)^\varepsilon}{\Gamma(1-\varepsilon)} \left\{ \left(\frac{s_{12}}{\mu^2}\right)^{-\varepsilon} \left[\mathcal{V}_{q\to qg}^{(0,1)} + \mathcal{V}_{g\to gg}^{(0,1)} \left(\frac{s_{23}}{s_{12}+s_{23}}\right) \right] + \right. \\ \left. + \left(\frac{s_{23}}{\mu^2}\right)^{-\varepsilon} \left[\mathcal{V}_{q\to qg}^{(0,1)} + \mathcal{V}_{g\to gg}^{(0,1)} \left(\frac{s_{12}}{s_{12}+s_{23}}\right) \right] \right\}. \qquad (5.56)$$

The explicit forms of the functions $\mathcal{V}_{q\to qg}^{(0,1)}$ and $\mathcal{V}_{q\to qg}^{(0,1)}$ are given in eq. (5.62). The subtraction term $d\alpha_n^{(1,1)}$ is written as a sum over topologies:

$$d\alpha_n^{(1,1)} = \sum_{\text{topologies } T} \mathcal{D}_n^{(1,1)}(T)\,. \qquad (5.57)$$

The relevant topologies for the leading N_c contributions to $e^+e^- \to qg g\bar{q}$ are shown in figure 8. It is sufficient to consider the topology $T_a^{(1,1)}$. The topology $T_b^{(1,1)}$ is related by symmetry to $T_a^{(1,1)}$. The subtraction term is given by

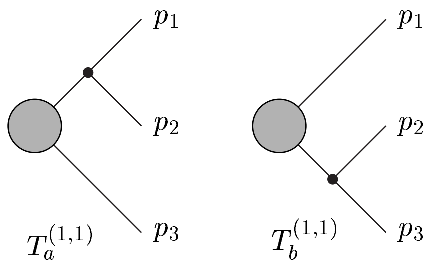

Figure 8: Splitting topologies for the subtraction terms for the one-loop amplitude with one unresolved parton.

$$\mathcal{D}_n^{(1,1)}(T) = \frac{c}{8\pi^2} \left[\left\langle A_n^{(0)} \right| \mathcal{P}^{(0,1)}(T) \left| A_n^{(1)} \right\rangle + \left\langle A_n^{(1)} \right| \mathcal{P}^{(0,1)}(T) \left| A_n^{(0)} \right\rangle + \right. \\ \left. + \left\langle A_n^{(0)} \right| \mathcal{P}^{(1,1)}(T) \left| A_n^{(0)} \right\rangle \right] d\phi_{n+1}\,.$$

Here, in the first and second term the NLO splitting function $\mathcal{P}^{(0,1)}$ is sandwiched between $A_n^{(0)}$ and $A_n^{(1)}$. The third term contains as a new structure the one-loop splitting function $\mathcal{P}^{(1,1)}$. It is convenient to write

$$\mathcal{P}^{(1,1)}(T) = \mathcal{P}_{(1,0)}^{(1,1)}(T) + \mathcal{P}_{(0,1)}^{(1,1)}(T)\,, \qquad (5.58)$$

where $\mathcal{P}_{(1,0)}^{(1,1)}$ is (part of) the approxiamation to $d\sigma_{n+1}^{(1)}$ and $\mathcal{P}_{(0,1)}^{(1,1)}$ is an approxiamation to $d\alpha_{n+1}^{(0,1)}$. The splitting functions read:

$$\begin{aligned}
\mathcal{P}^{(1,1)}_{(1,0)\,q\to qg}\left(T^{(1,1)}_a\right) &= S_\varepsilon^{-1} c_\Gamma \left(\frac{-s_{123}}{\mu^2}\right)^{-\varepsilon} y^{-\varepsilon} \\
&\quad \times \left\{ g_1(y,z)\, \mathcal{P}^{(0,1)}_{q\to qg} + g_2 \frac{2}{s_{123}} \frac{1}{y} \not{p}_e \left[1-\rho\varepsilon(1-y)(1-z)\right] \right\} - \\
&\quad - \frac{11}{6\varepsilon} \mathcal{P}^{(0,1)}_{q\to qg} + \text{c.c.}, \\
\mathcal{P}^{(1,1)}_{(0,1)\,q\to qg}\left(T^{(1,1)}_a\right) &= S_\varepsilon^{-1} \frac{(4\pi)^\varepsilon}{\Gamma(1-\varepsilon)} \left(\frac{s_{123}}{\mu^2}\right)^{-\varepsilon} \times \\
&\quad \times \left\{ y^{-\varepsilon} \left[\mathcal{V}^{(0,1)}_{q\to qg} + \mathcal{V}^{(0,1)}_{g\to gg}\left(\frac{(1-y)(1-z)}{1-z(1-y)}\right)\right] + \right. \\
&\quad \left. + (1-y)^{-\varepsilon}(1-z)^{-\varepsilon} \left[\mathcal{V}^{(0,1)}_{q\to qg} + \mathcal{V}^{(0,1)}_{g\to gg}\left(\frac{y}{1-z(1-y)}\right)\right] \right\} \mathcal{P}^{(0,1)}_{q\to qg}\,.
\end{aligned} \tag{5.59}$$

The variables y and z are given by

$$y = \frac{s_{12}}{s_{123}}\,, \qquad z = \frac{s_{13}}{s_{13}+s_{23}}\,. \tag{5.60}$$

The functions g_1 and g_2 are given by

$$\begin{aligned}
g_1(y,z) &= -\frac{1}{\varepsilon^2}\left[\Gamma(1+\varepsilon)\Gamma(1-\varepsilon)\left(\frac{z}{1-z}\right)^{\varepsilon} + \right. \\
&\quad \left. + 1 - (1-y)^\varepsilon z^\varepsilon\, {}_2F_1\left(\varepsilon,\varepsilon,1+\varepsilon;(1-y)(1-z)\right)\right], \\
g_2 &= \frac{1-\rho\varepsilon}{2(1-\varepsilon)(1-2\varepsilon)}\,.
\end{aligned} \tag{5.61}$$

The sum of the integrated dipoles reads

$$\begin{aligned}
\mathcal{V}^{(0,1)}_{q\to qg} + \mathcal{V}^{(0,1)}_{g\to gg}(r) &= \frac{\Gamma(-\varepsilon)\Gamma(-2\varepsilon)\Gamma(1-\varepsilon)^2}{\Gamma(1-2\varepsilon)\Gamma(1-3\varepsilon)} \times \\
&\quad \times \left\{ 4 + \frac{8\varepsilon}{1-3\varepsilon} - 2(1-\rho\varepsilon)\frac{\varepsilon(1-\varepsilon)}{(1-3\varepsilon)(2-3\varepsilon)} - \frac{4}{3}\frac{\varepsilon(1-\varepsilon)}{(1-3\varepsilon)(2-3\varepsilon)} r \right\}.
\end{aligned} \tag{5.62}$$

The amputated amplitudes are evaluated with the momenta

$$\begin{aligned}
p_e &= p_1 + p_2 - \frac{y}{1-y} p_3\,, \\
p_s &= \frac{1}{1-y} p_3\,.
\end{aligned} \tag{5.63}$$

For the numerical integration over the $(n+1)$-parton phase space we need the ε-expansion of the subtraction terms:

$$\begin{aligned}
\left\langle A^{(0)}_n \middle| \mathcal{P}^{(1,1)}(T) \middle| A^{(0)}_n \right\rangle &= A^{(0)\,*}_2 A^{(0)}_2 \times \\
&\quad \times \frac{2}{s_{12}} \left\{ \left[\frac{2}{\varepsilon^2} + (3-2L)\frac{1}{\varepsilon} + 2\,\mathrm{Li}_2\left((1-y)(1-z)\right) + 2\left(\ln(1-y)\right)^2 + \right.\right. \\
&\quad + 2\ln(1-y)\ln(z) + 2\ln(1-y)\ln(1-z) -
\end{aligned}$$

$$
\begin{aligned}
& - 2\ln(y)\ln(1-z) + 2\ln(z)\ln(1-z) - 2\ln(y)\ln(1-y) - \\
& - \frac{3}{2}\pi^2 - \frac{19}{6}\ln(1-z) - \frac{19}{6}\ln(1-y) - \frac{7}{2}\ln(y) + \frac{58}{3} \\
& + \frac{1}{3}\frac{(1-y)(1-z)}{1-z(1-y)}\left(\ln(y) - \ln(1-y) - \ln(1-z)\right) + \\
& + L^2 - \frac{20}{3}L + \rho\Bigg]\left[\frac{2}{1-z(1-y)} - 2 + (1-y)(1-z)\right] + \\
& + \left[-\frac{2}{\varepsilon} - 3 + 2L\right]\rho(1-y)(1-z) + 1\Bigg\} + O(\varepsilon)\,. \qquad (5.64)
\end{aligned}
$$

Here, $L = \ln(s_{123}/\mu^2)$. The ε-expansion of the insertion of $\mathcal{P}^{(0,1)}$ into the interference term with the loop amplitude $A_2^{(1)}$ is given by

$$
\begin{aligned}
\left\langle A_n^{(0)}\middle|\mathcal{P}^{(0,1)}(T)\middle|A_n^{(1)}\right\rangle + \left\langle A_n^{(1)}\middle|\mathcal{P}^{(0,1)}(T)\middle|A_n^{(0)}\right\rangle = \; & A_2^{(0)\,*}\,A_2^{(0)}\,\frac{2}{s_{12}} \times \\
& \times\Bigg\{\left[-\frac{2}{\varepsilon^2}+(-3+2L)\frac{1}{\varepsilon}-8+3L-L^2+\frac{7}{6}\pi^2\right] \times \\
& \quad \times\left[\frac{2}{1-z(1-y)} - 2 + (1-y)(1-z)\right] + \\
& \quad + \left[\frac{2}{\varepsilon} + 3 - 2L\right]\rho(1-y)(1-z)\Bigg\} + \\
& + O(\varepsilon)\,. \qquad (5.65)
\end{aligned}
$$

We see that the poles in ε cancel in the sum of the two contributions, as it should be.

6. Cancellations

In this section I discuss how singularities cancel between different terms for the various singular limits. The most intricate part is the one which is integrated over the $n+2$ parton phase space:

$$
\int \left(\mathcal{O}_{n+2}\,d\sigma_{n+2}^{(0)} - \mathcal{O}_{n+1}\circ d\alpha_{n+1}^{(0,1)} - \mathcal{O}_n\circ d\alpha_n^{(0,2)}\right), \qquad (6.1)
$$

where

$$
d\alpha_{n+1}^{(0,1)} = \sum_{T=T_a^{(1)},\dots,T_d^{(1)}} \mathcal{D}_{n+1}^{(0,1)}(T)\,, \qquad d\alpha_n^{(0,2)} = \sum_{T=T_a^{(2)},\dots,T_h^{(2)}} \mathcal{D}_n^{(0,2)}(T)\,. \qquad (6.2)
$$

Although individual terms in the integrand of eq. (6.1) are singular, the singularities cancel in the sum and the integrand is integrable. Therefore the integration can be performed by Monte Carlo methods. It is instructive to see in detail, how the cancellations of singularities take place. Critical region in phase space are regions where one or two particles are unresolved. Double unresolved contributions can be divided into the following classes:

a) Two pairs of separately collinear particles.

b) Three particles collinear.

c) Two particles collinear and a third soft particle.

d) Two soft particles.

e) Coplanar degeneracy.

The last item occurs only in the subtraction terms and is an artefact of our choice for the NLO subtraction terms. It is included here for completeness. The single unresolved contributions are divided into two classes:

f) Two particles collinear.

g) One soft particle.

For the leading-colour contributions to $e^+e^- \to qg g\bar{q}$ it is sufficient to check the following cases: Particles 1 and 2 collinear and particles 3 and 4 collinear (case a); particles 1, 2 and 3 collinear (case b); particles 1 and 2 collinear and particle 3 soft (case c); particles 2 and 3 soft (case d); the sum of the four-momenta of particles 2 and 3 linear dependent with the four-momenta of particles 1 and 4 (case e); particles 1 and 2 collinear (case f); particles 2 and 3 collinear (also case f); particle 2 soft (case g). All other cases are related by symmetry to the ones above.

All limits have been checked using the symbolic algebraic manipulation program "FORM" [47].

6.1 Double unresolved regions

In the double unresolved cases, the singularities in the NLO subtraction terms $d\alpha^{(0,1)}_{n+1}$ are cancelled by the subtraction terms $d\alpha^{(0,2)}_{(0,1)\,n}$. In particular we always have

$$\begin{aligned}
\mathcal{O}_{n+1}\,\mathcal{D}^{(0,1)}_{n+1}(T^{(1)}_a) - \mathcal{O}_n\,\mathcal{D}^{(0,2)}_{(0,1)\,n}(T^{(2)}_b) - \mathcal{O}_n\,\mathcal{D}^{(0,2)}_{(0,1)\,n}(T^{(2)}_c) &= \text{integrable}\,,\\
\mathcal{O}_{n+1}\,\mathcal{D}^{(0,1)}_{n+1}(T^{(1)}_b) - \mathcal{O}_n\,\mathcal{D}^{(0,2)}_{(0,1)\,n}(T^{(2)}_d) - \mathcal{O}_n\,\mathcal{D}^{(0,2)}_{(0,1)\,n}(T^{(2)}_h) &= \text{integrable}\,,\\
\mathcal{O}_{n+1}\,\mathcal{D}^{(0,1)}_{n+1}(T^{(1)}_c) - \mathcal{O}_n\,\mathcal{D}^{(0,2)}_{(0,1)\,n}(T^{(2)}_e) - \mathcal{O}_n\,\mathcal{D}^{(0,2)}_{(0,1)\,n}(T^{(2)}_g) &= \text{integrable}\,,\\
\mathcal{O}_{n+1}\,\mathcal{D}^{(0,1)}_{n+1}(T^{(1)}_d) - \mathcal{O}_n\,\mathcal{D}^{(0,2)}_{(0,1)\,n}(T^{(2)}_a) - \mathcal{O}_n\,\mathcal{D}^{(0,2)}_{(0,1)\,n}(T^{(2)}_f) &= \text{integrable}\,,
\end{aligned} \tag{6.3}$$

for all double unresolved cases. Here, "integrable" stands for "integrable over the double unresolved region". It remains to discuss the cancellations between $d\sigma^{(0)}_{n+2}$ and $d\alpha^{(0,2)}_{(0,0)\,n}$ in the double unresolved regions. It should be noted that in specific double unresolved limits some subtraction terms in eq. (6.3) may be integrable by themselves. This will also be briefly discussed below case by case.

6.1.1 Case 1 and 2 collinear and 3 and 4 collinear.

This case corresponds to the emission of two independent pairs of collinear particles ($p_1||p_2$ and $p_3||p_4$). For each pair of collinear particles we write the corresponding momenta according to eq. (4.13). We are interested in contributions, which scale in the double collinear limit as $|k_\perp|^{-4}$. Cancellations of these terms occurs in the following combination:

$$\mathcal{O}_{n+2}\,d\sigma^{(0)}_{n+2} - \mathcal{O}_n\,\mathcal{D}^{(0,2)}_{(0,0)\,n}\left(T^{(2)}_a\right) - \mathcal{O}_n\,\mathcal{D}^{(0,2)}_{(0,0)\,n}\left(T^{(2)}_b\right) = \text{integrable}\,. \tag{6.4}$$

For completeness I also state the cancellations for terms related to $d\alpha^{(0,1)}_{n+1}$:

$$\begin{aligned}
\mathcal{O}_{n+1}\,\mathcal{D}^{(0,1)}_{n+1}\left(T^{(1)}_a\right) - \mathcal{O}_n\,\mathcal{D}^{(0,2)}_{(0,1)\,n}\left(T^{(2)}_b\right) &= \text{integrable}\,,\\
\mathcal{O}_{n+1}\,\mathcal{D}^{(0,1)}_{n+1}\left(T^{(1)}_d\right) - \mathcal{O}_n\,\mathcal{D}^{(0,2)}_{(0,1)\,n}\left(T^{(2)}_a\right) &= \text{integrable}\,.
\end{aligned} \tag{6.5}$$

All other subtraction terms are separately integrable in this limit.

6.1.2 Case 1, 2 and 3 collinear.

This case corresponds to the triple collinear limit $(p_1||p_2||p_3)$. We parameterize the momenta of the three collinear particles according to eq. (4.23). We are interested in contributions, which scale in the double collinear limit as $|k_\perp|^{-4}$. Cancellations of these terms occurs in the following combination:

$$\mathcal{O}_{n+2}\,d\sigma^{(0)}_{n+2} - \mathcal{O}_n\,\mathcal{D}^{(0,2)}_{(0,0)\,n}(T^{(2)}_c) - \mathcal{O}_n\,\mathcal{D}^{(0,2)}_{(0,0)\,n}(T^{(2)}_d) - \mathcal{O}_n\,\mathcal{D}^{(0,2)}_{(0,0)\,n}(T^{(2)}_e) = \text{integrable}\,. \tag{6.6}$$

For completeness I also state the cancellations for terms related to $d\alpha^{(0,1)}_{n+1}$:

$$\begin{aligned}
\mathcal{O}_{n+1}\,\mathcal{D}^{(0,1)}_{n+1}(T^{(1)}_a) - \mathcal{O}_n\,\mathcal{D}^{(0,2)}_{(0,1)\,n}(T^{(2)}_c) &= \text{integrable}\,,\\
\mathcal{O}_{n+1}\,\mathcal{D}^{(0,1)}_{n+1}(T^{(1)}_b) - \mathcal{O}_n\,\mathcal{D}^{(0,2)}_{(0,1)\,n}(T^{(2)}_d) &= \text{integrable}\,,\\
\mathcal{O}_{n+1}\,\mathcal{D}^{(0,1)}_{n+1}(T^{(1)}_c) - \mathcal{O}_n\,\mathcal{D}^{(0,2)}_{(0,1)\,n}(T^{(2)}_e) &= \text{integrable}\,.
\end{aligned} \tag{6.7}$$

All other subtraction terms are separately integrable in this limit.

6.1.3 Case 1 and 2 collinear and 3 soft.

This case corresponds to the emission of a collinear pair $(p_1||p_2)$ together with a soft gluon (p_3). In this limit, the invariants s_{12}, s_{23}, s_{34} and s_{123} become small. We are interested in contributions, which have three of these four invariants in the denominator. Cancellations of these terms occurs in the following combination:

$$\mathcal{O}_{n+2}\,d\sigma^{(0)}_{n+2} - \sum_{T=T^{(2)}_a,\dots,T^{(2)}_g} \mathcal{O}_n\,\mathcal{D}^{(0,2)}_{(0,0)\,n}(T) = \text{integrable}\,. \tag{6.8}$$

For completeness I also state the cancellations for terms related to $d\alpha^{(0,1)}_{n+1}$:

$$\begin{aligned}
\mathcal{O}_{n+1}\,\mathcal{D}^{(0,1)}_{n+1}(T^{(1)}_a) - \mathcal{O}_n\,\mathcal{D}^{(0,2)}_{(0,1)\,n}(T^{(2)}_b) - \mathcal{O}_n\,\mathcal{D}^{(0,2)}_{(0,1)\,n}(T^{(2)}_c) &= \text{integrable}\,,\\
\mathcal{O}_{n+1}\,\mathcal{D}^{(0,1)}_{n+1}(T^{(1)}_c) - \mathcal{O}_n\,\mathcal{D}^{(0,2)}_{(0,1)\,n}(T^{(2)}_e) &= \text{integrable}\,,\\
\mathcal{O}_{n+1}\,\mathcal{D}^{(0,1)}_{n+1}(T^{(1)}_d) - \mathcal{O}_n\,\mathcal{D}^{(0,2)}_{(0,1)\,n}(T^{(2)}_a) &= \text{integrable}\,.
\end{aligned} \tag{6.9}$$

All other subtraction terms are separately integrable in this limit.

6.1.4 Case 2 and 3 soft

This case corresponds to the emission of two soft gluons (with momenta p_2 and p_3). If we scale the momenta of two soft particles as

$$p_2 \rightarrow \lambda p_2 \,, \qquad p_3 \rightarrow \lambda p_3, \tag{6.10}$$

we are interested in contributions, which scale in the double soft limit as λ^{-4}. Cancellations of these terms occurs in the following combination:

$$\mathcal{O}_{n+2}\, d\sigma_{n+2}^{(0)} - \sum_{T=T_c^{(2)},\ldots,T_h^{(2)}} \mathcal{O}_n\, \mathcal{D}_{(0,0)\,n}^{(0,2)}(T) = \text{integrable}\,. \tag{6.11}$$

For completeness I also state the cancellations for terms related to $d\alpha_{n+1}^{(0,1)}$:

$$\begin{aligned}
\mathcal{O}_{n+1}\, \mathcal{D}_{n+1}^{(0,1)}\left(T_b^{(1)}\right) - \mathcal{O}_n\, \mathcal{D}_{(0,1)\,n}^{(0,2)}\left(T_d^{(2)}\right) - \mathcal{O}_n\, \mathcal{D}_{(0,1)\,n}^{(0,2)}\left(T_h^{(2)}\right) &= \text{integrable}\,, \\
\mathcal{O}_{n+1}\, \mathcal{D}_{n+1}^{(0,1)}\left(T_c^{(1)}\right) - \mathcal{O}_n\, \mathcal{D}_{(0,1)\,n}^{(0,2)}\left(T_e^{(2)}\right) - \mathcal{O}_n\, \mathcal{D}_{(0,1)\,n}^{(0,2)}\left(T_g^{(2)}\right) &= \text{integrable}\,.
\end{aligned} \tag{6.12}$$

All other subtraction terms are separately integrable in this limit.

6.1.5 Coplanar case

For completeness I briefly discuss singularities occuring in coplanar configurations. The matrix element squared $d\sigma_{n+2}^{(0)}$ is not singular in this limit. This singularity occurs only in the subtraction terms. An example is given by the subtraction term $\mathcal{D}_{n+1}^{(0,1)}(T_b^{(1)})$. This subtraction term has a singularity in

$$2p_4 p_e^{(1)} = \frac{s_{123} s_{234} - s_{23} s_{1234}}{s_{123} - s_{23}}\,. \tag{6.13}$$

The numerator is a Gram determinant and can also be written as

$$s_{123} s_{234} - s_{23} s_{1234} = \langle 1 - | 2+3 | 4 - \rangle \langle 4 - | 2+3 | 1 - \rangle\,. \tag{6.14}$$

This expression vanishes, whenever the sum of the four-momenta $p_2 + p_3$ becomes linear dependent with p_1 and p_4. Therefore, $2p_4 p_e^{(1)}$ becomes small, whenever this condition is fullfilled and whenever $p_2 + p_3$ is not linear dependent with p_1. (In the later case also the denominator in eq. (6.13) vanishes.) This type of singularity occurs only in the subtraction terms and the cancellation takes place in the following combination:

$$\mathcal{O}_{n+1}\, \mathcal{D}_{n+1}^{(0,1)}\left(T_b^{(1)}\right) - \mathcal{O}_n\, \mathcal{D}_{(0,1)\,n}^{(0,2)}\left(T_h^{(2)}\right) = \text{integrable}\,. \tag{6.15}$$

6.2 Single unresolved regions

The expression in eq. (6.3) is integrated over the complete $(n+2)$-parton phase space. This implies in particular an integration over regions which correspond to single unresolved configurations. It is clear that in this case we cannot expect a cancellation of singularities between terms, which involve $\mathcal{O}_{n+2}$ and $\mathcal{O}_n$, e.g. similar to that what we had in the double unresolved case. In the single unresolved case the $(n+2)$-parton configuration used in the calculation of the observable $\mathcal{O}_{n+2}$ will go smoothly to a $(n+1)$-parton configuration. The numerical value for the observable $\mathcal{O}$ evaluated with this $(n+1)$-parton configuration will however in general have no relationship with the numerical value obtained from the evaluation with the n-parton configuration of the subtraction terms. A similar argument applies to the factorization of the matrix element. Both effects spoil a cancellation of the singularities. The cancellation of the singularities has to occur therefore either between terms involving $\mathcal{O}_{n+2}$ and $\mathcal{O}_{n+1}$, or between terms involving $\mathcal{O}_{n+1}$ and $\mathcal{O}_n$, or between terms involving only $\mathcal{O}_n$. In the following, "integrable" stands for "integrable over the single unresolved region".

6.2.1 Case 1 and 2 collinear

This case corresponds to the collinear limit $p_1||p_2$. We parameterize the momenta of the two collinear particles according to eq. (4.13). We are interested in contributions, which scale in the collinear limit as $|k_\perp|^{-2}$. A cancellations of these terms occurs in the following combination:

$$\mathcal{O}_{n+2}\, d\sigma^{(0)}_{n+2} - \mathcal{O}_{n+1}\, \mathcal{D}^{(0,1)}_{n+1}\left(T^{(1)}_a\right) = \text{integrable}\,. \tag{6.16}$$

This is just the combination which already occurs at NLO. At NNLO we also have to take into account the singularity in $\mathcal{D}^{(0,1)}_{n+1}(T^{(1)}_d)$. This one is cancelled in the combination

$$\mathcal{O}_{n+1}\, \mathcal{D}^{(0,1)}_{n+1}\left(T^{(1)}_d\right) - \mathcal{O}_n\, \mathcal{D}^{(0,2)}_{(0,1)\,n}\left(T^{(2)}_a\right) = \text{integrable}\,. \tag{6.17}$$

Finally, there is a third equation, describing the cancellations among the NNLO subtraction terms:

$$\mathcal{O}_n\, \mathcal{D}^{(0,2)}_{(0,0)\,n}\left(T^{(2)}_b\right) + \mathcal{O}_n\, \mathcal{D}^{(0,2)}_{(0,0)\,n}\left(T^{(2)}_c\right) - \mathcal{O}_n\, \mathcal{D}^{(0,2)}_{(0,1)\,n}\left(T^{(2)}_b\right) - \mathcal{O}_n\, \mathcal{D}^{(0,2)}_{(0,1)\,n}\left(T^{(2)}_c\right) = \text{integrable}\,. \tag{6.18}$$

All other subtraction terms are separately integrable in this limit. Note that the pieces

$$\mathcal{O}_n\, \mathcal{D}^{(0,2)}_{(0,0)\,n}\left(T^{(2)}_b\right) - \mathcal{O}_n\, \mathcal{D}^{(0,2)}_{(0,1)\,n}\left(T^{(2)}_b\right) \tag{6.19}$$

and

$$\mathcal{O}_n\, \mathcal{D}^{(0,2)}_{(0,0)\,n}(T^{(2)}_c) - \mathcal{O}_n\, \mathcal{D}^{(0,2)}_{(0,1)\,n}(T^{(2)}_c) \tag{6.20}$$

are with our choice of subtraction terms in general **not** integrable separately. Only the sum of the two pieces is integrable. In order for the cancellations to occur, the n-parton configuration of topology $T^{(2)}_b$ must approach in the limit $s_{12} \to 0$ the same configuration as the n-parton configuration of topology $T^{(2)}_c$. With a symmetric choice for the reconstruction functions in eq. (5.31) and eq. (5.38) the n-parton configurations for topologies $T^{(2)}_b$ and $T^{(2)}_c$ are identical for any value of s_{12} and this requirement is fullfilled trivially.

6.2.2 Case 2 and 3 collinear.

Here we discuss the case, where particles 2 and 3 become collinear. Again, we parameterize the momenta of the two collinear particles according to eq. (4.13). We are interested in contributions, which scale in the collinear limit as $|k_\perp|^{-2}$. The NLO relation for the cancellation of singularities reads:

$$\mathcal{O}_{n+2}\, d\sigma^{(0)}_{n+2} - \mathcal{O}_{n+1}\, \mathcal{D}^{(0,1)}_{n+1}\left(T^{(1)}_b\right) - \mathcal{O}_{n+1}\, \mathcal{D}^{(0,1)}_{n+1}\left(T^{(1)}_c\right) = \text{integrable}\,. \tag{6.21}$$

In addition, there is one relation among the subtraction terms with n-parton configurations:

$$\begin{aligned}
&\mathcal{O}_n\, \mathcal{D}^{(0,2)}_{(0,0)\,n}\left(T^{(2)}_d\right) + \mathcal{O}_n\, \mathcal{D}^{(0,2)}_{(0,0)\,n}\left(T^{(2)}_h\right) - \mathcal{O}_n\, \mathcal{D}^{(0,2)}_{(0,1)\,n}\left(T^{(2)}_d\right) - \mathcal{O}_n\, \mathcal{D}^{(0,2)}_{(0,1)\,n}\left(T^{(2)}_h\right) + \\
&+ \mathcal{O}_n\, \mathcal{D}^{(0,2)}_{(0,0)\,n}\left(T^{(2)}_e\right) + \mathcal{O}_n\, \mathcal{D}^{(0,2)}_{(0,0)\,n}\left(T^{(2)}_g\right) - \mathcal{O}_n\, \mathcal{D}^{(0,2)}_{(0,1)\,n}\left(T^{(2)}_e\right) - \mathcal{O}_n\, \mathcal{D}^{(0,2)}_{(0,1)\,n}\left(T^{(2)}_g\right) = \\
&\qquad = \text{integrable}\,.
\end{aligned} \tag{6.22}$$

All other subtraction terms are separately integrable in this limit. Again we have to require that the n-parton configurations of the topologies $T^{(2)}_d$, $T^{(2)}_e$, $T^{(2)}_g$ and $T^{(2)}_h$ all approach the same n-parton configuration in the limit $s_{23} \to 0$. With the choice for the reconstruction functions in eq. (5.45) and eq. (5.52) this requirement is fullfilled.

6.2.3 Case 2 soft

This case corresponds to the emission of a single soft gluon (particle 2). If we scale the momentum of the soft gluon as

$$p_2 \to \lambda p_2\,, \tag{6.23}$$

we are interested in contributions, which scale in the soft limit as λ^{-2}. Cancellations of these terms occurs in the following combination:

$$\mathcal{O}_{n+2}\, d\sigma^{(0)}_{n+2} - \mathcal{O}_{n+1}\, \mathcal{D}^{(0,1)}_{n+1}\left(T^{(1)}_a\right) - \mathcal{O}_{n+1}\, \mathcal{D}^{(0,1)}_{n+1}\left(T^{(1)}_b\right) = \text{integrable}\,. \tag{6.24}$$

This is the usual NLO relation. In addition, we have

$$\begin{aligned}
&\mathcal{O}_n\, \mathcal{D}^{(0,2)}_{(0,0)\,n}\left(T^{(2)}_b\right) + \mathcal{O}_n\, \mathcal{D}^{(0,2)}_{(0,0)\,n}\left(T^{(2)}_c\right) - \mathcal{O}_n\, \mathcal{D}^{(0,2)}_{(0,1)\,n}\left(T^{(2)}_b\right) - \mathcal{O}_n\, \mathcal{D}^{(0,2)}_{(0,1)\,n}\left(T^{(2)}_c\right) = \text{integrable}\,, \\
&\mathcal{O}_n\, \mathcal{D}^{(0,2)}_{(0,0)\,n}\left(T^{(2)}_d\right) + \mathcal{O}_n\, \mathcal{D}^{(0,2)}_{(0,0)\,n}\left(T^{(2)}_h\right) - \mathcal{O}_n\, \mathcal{D}^{(0,2)}_{(0,1)\,n}\left(T^{(2)}_d\right) - \mathcal{O}_n\, \mathcal{D}^{(0,2)}_{(0,1)\,n}\left(T^{(2)}_h\right) = \text{integrable}\,.
\end{aligned} \tag{6.25}$$

All other subtraction terms are separately integrable in this limit.

6.3 Cancellations for contributions with one virtual and one real unresolved parton

The cancellation mechanism for contributions with one virtual and one real unresolved parton are quite similar to those already encountered at NLO. The only difference is the functional form of the subtraction terms (as given in eq. (5.59)), which approach smoothly the appropriate singular limit (as given in eq. (4.22)).

7. Outlook and conclusions

In this paper I considered the extension of the subtraction method to next-to-next-to-leading order. Such a method is needed to extend fully differential perturbative calculations in quantum field theories from NLO to NNLO. As a particular example I considered the leading-colour contributions to $e^+e^- \rightarrow$ 2 jets and derived the subtraction terms at NNLO. The subtracted matrix elements can be integrated numerically over the appropriate phase space. The method presented here is general and not restricted to the example of $e^+e^- \rightarrow$ 2 jets. Subtraction terms for other splittings (like $g \rightarrow ggg$) and other kinematical configuartions (e.g. with partons in the initial state) can be worked out along the same lines.

The subtraction terms still have to be integrated analytically over the unresolved phase space. With the advance of integration techniques witnessed in the past years, this seems feasible. The completion of this program will open the door to fully differential NNLO Monte Carlo programs.

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